Modèles de Mumford-Shah pour la détection de structures fines en image

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UNIVERSITÉ D'ORLÉANS

ÉCOLE DOCTORALE ***MATHEMATIQUES, INFORMATIQUE, PHYSIQUE THEORIQUE ET INGENIERIE DES SYSTEMES***

MAPMO

THÈSE présentée par :

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soutenue le : **14 septembre 2015**

pour obtenir le grade de : **Docteur de l'université d'Orléans**

Discipline/ Spécialité : Mathématiques

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Remerciements

Je voudrais tout d'abord remercier Maïtine Bergounioux de m'avoir proposé ce travail de thèse ainsi que pour ses précieux conseils, sa patience et sa disponibilité. Merci à Dorin Bucur et Simon Masnou d'avoir bien voulu accepter d'être rapporteurs et à Guy Bouchitté, Antonin Chambolle et Luc Hillairet de participer au jury de cette thèse.

Je tiens à remercier toute l'équipe du MAPMO pour m'avoir offert un cadre stimulant de recherche. En particulier, j'ai pu profiter des séances de travail du Groupe Image, du Réseau Thématique Image Régional et du Groupe des doctorants. Un grand merci donc aux organisateurs Cécile Louchet, Laurent Delsol et Rémy Buffe.

À titre plus personnel, je tiens à remercier mes parents, ma famille et mes proches, en particulier Anaïs pour m'avoir patiemment écouté parler de sujets qui ne font pas partie de ses centres d'intérêts majeurs...

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Introduction

1.1 Le problème et sa modélisation

Cette thèse est une contribution au problème de détection de fines structures tubulaires dans une image 2D ou 3D. En particulier, nous sommes intéressés par l'extraction du réseau sanguin dans une angiographie obtenue par IRM du cerveau d'une souris. L'intensité en chaque point est alors donnée par une valeur unidimensionnelle.

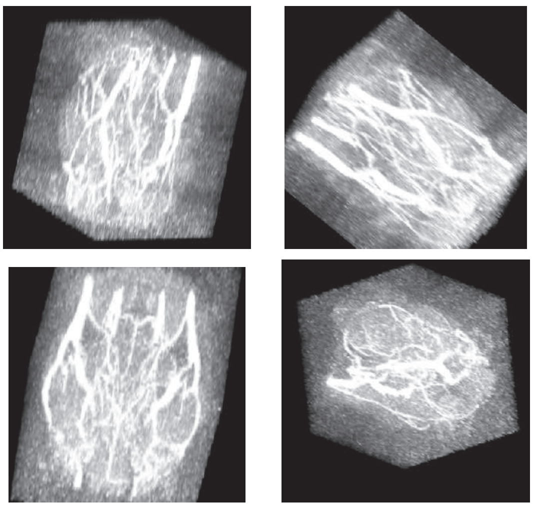


FIGURE 1.1.1 – Angiographie cérébrale d'une souris

Nous mettons en avant les caractéristiques de luminosité et géométriques qui permettent de distinguer le réseau sanguin du reste de l'image en s'appuyant sur une sélection de travaux déjà existants.

- **La luminosité.** La technique d'imagerie, l'angiographie par IRM, utilise l'injection d'un produit de contraste dans le réseau sanguin de l'animal. Cela a pour conséquence le réhaussement de la luminosité des vaisseaux. La différence de luminosité entre ceux-ci et leur voisinage est analysée pour construire une méthode de segmentation du réseau sanguin dans [BRL+04]. Cette méthode est limitée par l'inhomogénéité de l'intensité. Cependant, dans [CF05], les auteurs tirent parti du fait qu'à l'intérieur des vaisseaux sanguins l'homogénéité est plus forte qu'à l'extérieur pour améliorer cette technique.
- **Le bruit.** On considère comme bruit les pixels qui sont d'intensité comparable à celle du réseau sanguin mais qui n'appartiennent pas à un vaisseau. Dans [HFHM03], une étude statistique du bruit est réalisée afin d'élaborer une stratégie de segmentation. Cependant, du fait de la petite taille du cerveau d'une souris en comparaison avec la taille humaine, l'importance du bruit est accrue dans notre situation. On peut observer que pour les plus fines structures, le rayon tubulaire est comparable à la section d'un *atome* du bruit.
- **La géométrie.** La discrimination du réseau sanguin du reste de l'image tient aussi à sa forme spécifique. On caractérise la géométrie d'un vaisseau par son élongation, son rayon, sa courbure et par la présence d'une bifurcation. On peut cataloguer plusieurs types de modélisations géométriques.
  - *Paramétrisation explicite d'une surface.* Pour un vaisseau sanguin suffisamment important, on peut rechercher une paramétrisation explicite de sa surface. Cette approche conduit à des techniques de segmentation par contours actifs. Dans [dBvGVN03], les auteurs proposent d'adapter cette technique au cas d'une surface tubulaire. Ce type d'approche est adapté à un réseau dont le nombre et les positions approximatives des bifurcations seraient connues par avance.
  - *Paramétrisation linéique des vaisseaux sanguins.* Dans le cas où le rayon du tube varie peu, on peut représenter les vaisseaux sanguins par la ligne centrale du tube. Dans [LGFM06] et [BST05], les auteurs modélisent le réseau sanguin comme la réunion de courbes paramétrées, à l'aide respectivement de segments ou de B-splines. Les techniques dérivées de cette modélisation sont dépendantes de la courbure du réseau. Elles supposent que l'on en connaisse au préalable une borne supérieure.
- **Les vaisseaux comme chemins géodésiques.** Une modélisation des

vaisseaux comme plus court chemin, pour une certaine métrique, d'un point à un autre de l'image a été proposée dans [CK97]. Afin de résoudre le problème des croisements de deux vaisseaux dans une image 2D, dans [PPK09], les auteurs proposent d'augmenter l'espace 2D en considérant aussi les directions. L'espace ambiant sur lequel sont tracés les chemins est donc de dimension 4.

– **Modélisations hybrides.** Certaines modélisations combinent la luminosité et la géométrie.
  - *Les vaisseaux comme lignes de crêtes.* En considérant qu'un point de l'image est caractérisé par sa position et son intensité, dans [Blu62], l'auteur suggère de considérer une image comme une hypersurface de $\mathbb{R}^{d+1}$, où d est la dimension de l'image. Plus précisément, de ce point de vue, cette surface est le graphe d'une fonction. Un vaisseau sanguin apparaît alors comme une *ligne de crête* du graphe de cette fonction. L'épaisseur de la ligne de crête représente le rayon des tubes alors que sa hauteur représente son intensité lumineuse. Des méthodes de détection locale des lignes de crêtes ont été proposées dans [SSDE96], [PMG+96].
  - *Classification des pixels par ondelettes.* En gardant le point de vue précédent, où une image est modélisée par une fonction, on peut penser à exploiter la décomposition en ondelettes de cette fonction pour caractériser les tubes. Dans [JCJ03], les auteurs représentent chaque pixel par un vecteur comprenant les mesures prises à différentes échelles d'une transformée en ondelettes. Ils utilisent ensuite cette classification pour déterminer si un pixel appartient ou pas à un vaisseau sanguin.

– **Modélisation variationnelle.** Comme précédemment, les modèles variationnels considèrent une image comme une fonction. Il s'agit, dans ce cas, de trouver une fonction qui minimise une certaine énergie qui mesure la distance à un *idéal*. De tels modèles dans le cas spécifiques de la détection d'ensembles de co-dimension 2 (ce qui est le cas d'un filament) ont été étudiés dans [AABF06, ABFG12]. Le modèle autour duquel s'articule cette thèse est celui de Mumford-Shah [MS89]. On utilise une approximation de cette énergie par champs de phase qui a été introduite dans [Mod87] pour le cas binaire et dans [AT90] dans le cadre général.

1.2 Structure du mémoire

On peut distinguer trois étapes dans cette thèse.

1. Dans une première approche, nous choisissons le cas où l'image initiale est donnée par un tube d'intensité et de rayon constants et nous considérons comme énergie l'approximation de la fonctionnelle de Mumford-Shah déduite de [Mod87]. Nous étudions le problème de minimisation de cette énergie lorsque celle-ci est restreinte à un sous-espace de fonctions qui déterminent une géométrie fixée. Plus précisément, cet espace est égal aux fonctions dont le support est contenu dans un tube donné et qui satisfont une symétrie cylindrique par rapport au centre de ce tube. En raison de cette symétrie, le problème se réduit à la dimension un. Un tube est alors caractérisé par son profil sur une section. Nous démontrons alors que ce problème admet une unique solution qui est caractérisée comme l'unique solution d'une équation différentielle. Nous exploitons cela pour décrire ce profil en fonction des paramètres du modèle et de l'épaisseur, de la longueur et de la courbure du tube.
2. La deuxième étape a pour but la modification du modèle de Mumford-Shah. Afin de se faciliter la tâche, nous conservons l'hypothèse simplificatrice de la première étape : on suppose que l'image est *bimodale*, c'est-à-dire que l'histogramme des intensités comprend deux modes. Le mode le plus haut correspond à la luminosité des vaisseaux sanguins et l'autre à l'intensité du fond. Nous recherchons alors une solution dans le sous-espace des fonctions binaires. Afin de favoriser la détection de fines structures tubulaires, le principe est d'ajouter comme inconnue du problème une métrique Riemannienne qui admet en tout point une direction propre dominante. Cette direction dominante représente l'orientation du tube et la valeur propre associée correspond à son élongation. La pénalisation des contours donnée par le modèle initial était *isotrope* (indépendante de la direction) et *homogène* (indépendante de la position). On la remplace alors par un terme *anisotrope* et *inhomogène*. En s'appuyant sur les travaux de [AFP00, Bou90] sur les fonctionnelles anisotropes, nous montrons que le problème de minimisation associé admet une solution. D'autre part, nous proposons et nous démontrons un résultat d'approximation de notre énergie par Γ-convergence. Cette approximation permet de donner une formulation dont la résolution numérique est plus facile à mettre en oeuvre.

3. La dernière étape de la thèse est la suite logique des deux étapes précédentes. Nous ne faisons plus d'hypothèse simplificatrice : il s'agit de proposer un modèle de détection des tubes dans le cadre général. Le résultat obtenu dans l'étape précédente nous fournit directement la modification à apporter au modèle initial. Le travail consiste d'abord à montrer que le problème est bien posé, c'est-à-dire qu'il admet une solution. Pour le modèle initial, le résultat avait été obtenu dans [GCL89] en montrant que le problème était équivalent à une formulation relaxée dans l'espace des fonctions spéciales à variations bornées. La preuve de ce résultat utilisait le théorème de compacité d'Ambrosio [Amb89] et un lemme de décroissance pour les minimiseurs de la fonctionnelle de Mumford-Shah. Dans notre cas, la fonctionnelle de Mumford-Shah étant perturbée par le terme anisotrope et inhomogène, nous utilisons un résultat récent [BL13] généralisant celui de De Giorgi-Carriero-Leaci à une classe de fonctions plus large qui inclut le cas des minimiseurs de notre fonctionnelle (*almost quasi-minimizers of free boundary problem*). Nous montrons ainsi que notre problème est bien posé et qu'il est équivalent à sa formulation relaxée dans l'espace des fonctions spéciales à variation bornée. L'étape suivante consiste à approcher notre énergie par Γ-convergence sur le modèle du résultat obtenu par Ambrosio-Tortorelli [AT90]. Ce travail utilisait une approximation de la mesure de Hausdorff $\mathcal{H}^{n-1}$ par un contenu de Minkowski. Dans notre situation, il s'agit d'introduire une définition adaptée à notre cadre anisotrope et inhomogène et ensuite de montrer qu'on a bien le même résultat d'approximation. Ce travail a été réalisé dans [BPV96, CLL14] dans le cadre binaire. Il nous faut donc étendre ce résultat à une classe d'ensembles plus large. L'outil principal de cette étape est le résultat de régularité (*Ahlfors regularity*) démontré dans [BL13]. Nous pouvons alors démontrer le résultat de Γ-convergence par des techniques de découpage par tranches (*slicing*) tirées de [AFP00].

Chapitre 1

Un problème de minimisation sous contraintes

1.1 Résumé

Ce travail a pour motivation un problème proposé par des biologistes du Centre de Biophysique Moléculaire (CBM) d'Orléans [1]. Dans le cadre de l'étude de maladies vasculaires, ceux-ci disposent d'angiographies cérébrales de souris obtenues par Imagerie à Résonance Magnétique (IRM). Le sujet de leur recherche est l'étude des fines structures du réseau sanguin. Leur souhait est de disposer d'une méthode de segmentation des fines structures tubulaires de leurs images. Cependant, du fait de la taille de l'animal, la segmentation des petites échelles du réseau sanguin est d'autant plus difficile que celles-ci se trouvent « noyées » dans un bruit dont la taille et l'intensité sont comparables.

Nous choisissons de poser le problème dans le cadre du calcul des variations. Dans cette approche, la solution d'un problème est définie comme l'état du système qui minimise une certaine énergie. Une difficulté majeure de cette approche est de déterminer quelle énergie modélise le problème. Cependant, proposer une énergie suppose au préalable une bonne connaissance du système que l'on souhaite modéliser. C'est pourquoi, nous différons cette étape et nous préférons, dans cette première partie de la thèse, faire un travail d'analyse. Nous prendrons donc un modèle déjà existant, l'énergie de Mumford-Shah, et nous étudions une solution de ce modèle dans le cas d'une

1. http://cbm.cnrs-orleans.fr

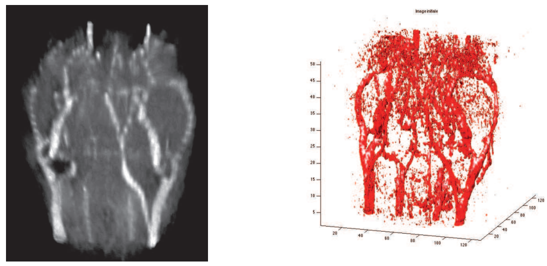


FIGURE 1.1.1 – Angiographie et seuillage à 60% de l'intensité maximale

image constituée de fins tubes.

D'autre part, nous nous concentrons ici sur les aspects géométriques du problème. Nous simplifions alors le problème en supposant que l'image à segmenter est celle obtenue après seuillage. C'est-à-dire que nous nous plaçons dans un cadre binaire. Nous voulons ainsi faire abstraction des problèmes liés aux inhomogénéités de luminosité. Le but est alors, pour un jeu de paramètres du modèle fixés, de faire apparaître la dépendance de la solution par rapport aux données géométriques de l'image comme le rayon, la longueur et la courbure des tubes.

On note n la dimension ($n = 2$ ou $n = 3$) et $\Omega \subset \mathbb{R}^n$ le domaine de l'image. En tout point $x \in \Omega$ est associée une intensité (normalisée) $g(x) \in [0, 1] \subset \mathbb{R}$. L'énergie de Mumford-Shah associée à g est définie par

$$\mathcal{E}(u, K) = \int_\Omega (u - g)^2 \mathrm{d}x + \beta \mathcal{H}^{n-1}(K) + \gamma \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}x,$$

où $\mathcal{H}^{n-1}$ est la mesure de Hausdorff de dimension $n - 1$, K est un compact de Ω de codimension 1, $u \in \mathrm{W}^{1,2}(\Omega \setminus K)$ et β, γ sont deux paramètres strictement positifs. Ici $W^{s,p}(\Omega)$ est l'espace de Sobolev habituel (voir par exemple [AF03]). Cette énergie dépend donc de deux variables, une fonction u et un ensemble K de dimension $n - 1$. En reprenant la modélisation décrite dans la section 1.1, le graphe de g représente un *paysage*, où K représente l'ensemble des lignes de crête et le graphe de u représente le paysage où on a lissé les petites irrégularités locales.

Si on fait l'hypothèse simplificatrice de binarité, on cherche alors u sous la forme d'une fonction indicatrice $u = \mathbf{1}_A$ et $K = \partial A$. La restriction de l'énergie de Mumford-Shah aux fonctions indicatrices, toujours notée $\mathcal{E}$, se

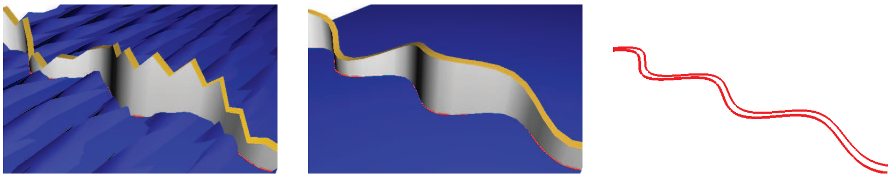


FIGURE 1.1.2 – Graphe de g et décomposition (u, K)

réduit alors à

$$\mathcal{E}(\mathbf{1}_A) = \int_\Omega (\mathbf{1}_A - g)^2 \mathrm{d}x + \beta \mathcal{H}^{n-1}(\partial A).$$

Le premier terme de cette énergie est une mesure *volumique* (la mesure de Lebesgue de dimension n) alors que le second est une mesure *surfacique* (la mesure de hausdorff de dimension $n-1$). Afin de disposer d'une énergie ne dépendant de qu'un seul type de mesure, on introduit une approximation de cette énergie

$$E_\varepsilon(p) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_\Omega \left(9\varepsilon |\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon} \right) \mathrm{d}x$$

où $p \in \mathrm{W}^{1,2}(\Omega; [0;1])$. Il est montré dans [Mod87] que la famille de fonctionnelles $(E_\varepsilon)_\varepsilon$ Γ-converge vers $\mathcal{E}$ pour $\varepsilon \to 0^+$. En particulier, si $(\varepsilon_k)_k$ converge vers 0^+ et si p_k est un minimiseur de E_{ε_k} pour tout k, alors la suite $(p_k)_k$ admet comme point d'accumulation un minimiseur de $\mathcal{E}$.

Nous renvoyons à [Mas93] (par exemple) et à la bibliographie de l'article qui suit pour une définition et des propriétés fondamentales de la Γ-convergence.

On modélise un tube d'intensité et de rayon constants A_α comme l'ensemble des points à une distance inférieure à $\alpha > 0$ d'une courbe Γ. On pose alors $g = \mathbf{1}_{A_\alpha}$.

On restreint alors E_ε à l'ensemble $\mathcal{F}_\alpha$ des fonctions dont le support est inclus dans A_α et qui sont à symétrie *tubulaire* (Définition 3.2.). On démontre alors que la minimisation de E_ε admet une unique solution (Théorème 3.1. et Théorème 3.3.). Cette unique solution est caractérisée en dimension 2 et 3 par une équation unidimensionnelle (Théorème 3.6.) et de cette équation nous déduisons des inégalités sur le *profil* de la solution (Théorème 3.7.).

Ces résultats ont fait l'objet de l'article suivant présenté dans la section suivante.

FIGURE 1.1.3 – Voisinage tubulaire de Γ

1.2 Article [BV14]

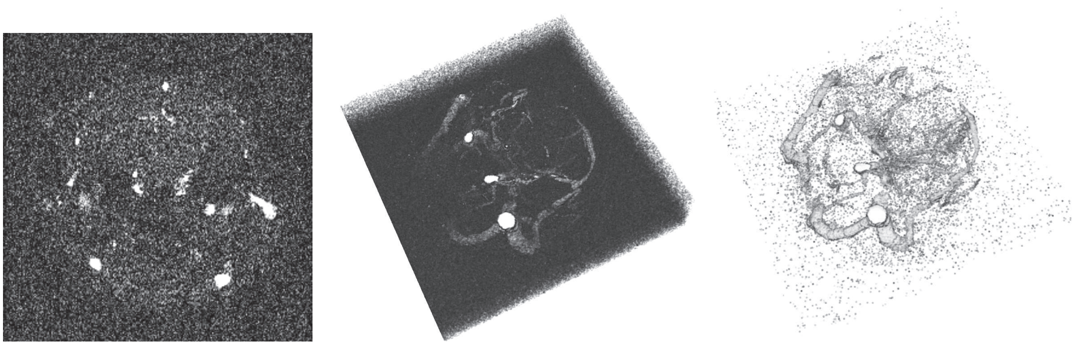
PARAMETER SELECTION IN A MUMFORD–SHAH GEOMETRICAL MODEL FOR THE DETECTION OF THIN STRUCTURES

MAÏTINE BERGOUNIOUX & DAVID VICENTE

Abstract. We present a variational model to perform the segmentation of thin structures in MRI images (namely codimension 1 objects). It is based on the classical Mumford-Shah functional and we have added geometrical priors as constraints. We precisely describe the structure model (that we call tubes). We give existence, uniqueness and regularity results for the solution to the optimization problem. The keypoint is the fact that 2D/3D problems are equivalent to 1D ones. This gives hints to perform an automatic parameter tuning for numerical purpose.

1. Introduction

The detection of blood vessels and the complete reconstruction of the network is one of the most challenging problems in biological image processing. Some angiography images are not very noisy and the identification of the network can be done by proven methods that we mention below. However, in some cases, the images are very noisy and undersampled. This is the case for example of angiographic MRI brain network mouse [1]. Even if the magnetic fields are high, the images are sub-sampled and low contrasted due to the smallness of the observed area. On the other hand the nature of the structure to identify (filaments of codimension 1) requires the development of models to identify objects of null measure.



(a) 2D slice (256 x 256) (b) 3D view (54 slices) (c) 3D view (54 slices)

Figure 1.1. Mouse brain MRI image with manual threshold

Several approaches have been made to overcome this difficulty both from the point of view in the theoretical aspect (models) and numerics (how to approach and / or discretize such structures). There are, to our knowledge, few models providing a satisfactory answer to these problems. In [15] many vessel extraction techniques and algorithms are

Date: May 21, 2015.

[1]We thank the laboratory CBM in Orléans for the images

presented. Vessel segmentation algorithms and techniques are divided into six main categories: pattern recognition techniques, model-based, tracking-based , artificial intelligence-based, neural network-based and tube-like object detection approaches. One can find a review of 3D vessel segmentation techniques in [17]. Recently, Péchaud and al. [21] have presented a method to extract a network of vessels centerlines from a medical image. They use both geodesic based methods and tracking methods in a 4D framework. Rouchdy and Cohen [24] use a geodesic voting method to consider the problem. In this paper, we have decided to use a variational model. Such models have been investigated by Aubert and al [6, 7, 8, 9, 14] especially in the detection of points in 2D images. One important tool is the capacity theory [1].

We present here a modified Mumford-Shah model. This model [20, 19] is a well known segmentation model whose approximation has been studied by many people (see for example [12, 13, 18, 16, 22]. We use the Ambrosio-Tortorelli [3] approach and consider an approximate model that Γ-converges to the original one [25]. As in [4] we add prior information on the objects but we involve this prior in the constraints rather than in the cost functional.

The paper is organized as follows. We first present the exact model we use and the approximated one. However, these models are not convex and we have a lack of uniqueness. Therefore we consider geometrical constraints in the objects that have to be what we define as *tubes* of small diameter. Section 3 is devoted to the description of 2D and 3D tubes. In the last section we prove that the problem reduces to a 1D problem and we give some qualitative properties of the (unique) solution. This allows to get an automatic parameter selection: indeed variational models are efficient but the parameter tuning is a major challenge. We end the paper with an appendix including the most technical proofs.

2. A Mumford- Shah type model

2.1. **The *exact* model.** Let Ω be an open bounded subset of $\mathbb{R}^N$ $(N = 2, 3)$ smooth enough ($\mathcal{C}^1$ for example). Let be $g\colon \Omega \to [0,1]$ the (normalized) observed image, $g(x)$ corresponds to the gray-scale intensity at point x. The model we study is derived from the Mumford-Shah one [20] that we briefly recall : we look for a pair (u, K) where $K \subset \Omega$ is the set of discontinuities of g and u is a regular function defined on $\Omega \setminus K$. This representation must minimize the following energy:

$$\mathcal{E}(u, K) = \frac{1}{2}\int_{\Omega\setminus K}(u-g)^2\,dx + \beta\mathcal{H}^{n-1}(K) + \gamma\int_{\Omega\setminus K}|\nabla u|^2\,dx, \tag{2.1}$$

where $\beta,\ \gamma > 0$ and $\mathcal{H}^{N-1}(K)$ is the Hausdorff measure of the $N-1$ dimensional set K. The first term is a fitting data term and the second ones penalizes the length (if $N = 2$) or area (if $N = 3$) of the discontinuity set. The last term penalizes u variations.

We want to split the image in two sub-domains A and $\Omega\setminus A$. So we only consider binary functions $u = \chi_A$ where:

$$\chi_A(x) = \begin{cases} 1 & \text{si } x \in A\ , \\ 0 & \text{si } x \in \Omega \setminus A. \end{cases}$$

As χ_A is constant outside its jump set, $\nabla\chi_A(x) = 0$ for any $x \in \Omega\setminus\partial A$ and the energy we have to minimize writes :

$$\mathcal{E}(\chi_A, \partial A) = \frac{1}{2}\int_{\Omega}(\chi_A - g)^2\,dx + \beta\mathcal{H}^{n-1}(\partial A). \tag{2.2}$$

To describe the jumps of the function u, the most suitable space is the space of of functions with bounded variation $BV(\Omega)$. We recall the definition and the main properties of this space (see [2, 5, 10] for example), defined by

$$BV(\Omega) = \{u \in L^1(\Omega) \mid \Phi_1(u) < +\infty\},$$

where

$$\Phi_1(u) := \sup\left\{\int_\Omega u(x)\,\mathrm{div}\,\xi(x)\,dx \mid \xi \in \mathcal{C}^1_c(\Omega),\ \|\xi\|_\infty \le 1\right\}. \tag{2.3}$$

The space $BV(\Omega)$, endowed with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1} + \Phi_1(u)$, is a Banach space. The derivative in the sense of distributions of every $u \in BV(\Omega)$ is a bounded Radon measure, denoted Du, and $\Phi_1(u) = \int_\Omega |Du|$ is the total variation of u. We next recall standard properties of functions of bounded variation .

Proposition 2.1. *Let Ω be an open subset of $\mathbb{R}^N$ with Lipschitz boundary.*

(1) *For every $u \in BV(\Omega)$, the Radon measure Du can be decomposed into $Du = \nabla u\,dx + D^s u$, where $\nabla u\,dx$ is the absolutely continuous part of Du with respect of the Lebesgue measure and $D^s u$ is the singular part.*
(2) *The mapping $u \mapsto \Phi_1(u)$ is lower semi-continuous from $BV(\Omega)$ to $\mathbb{R}^+$ for the $L^1(\Omega)$ topology.*
(3) *$BV(\Omega) \subset L^\sigma(\Omega)$ with continuous embedding, for $\sigma \in [1, \frac{N}{N-1}]$ ($N \neq 1$).*
(4) *$BV(\Omega) \subset L^\sigma(\Omega)$ with compact embedding, for $\sigma \in [1, \frac{N}{N-1})$ ($N \neq 1$).*

The singular part $D^s u$ of the derivative has a Cantor component. The functions we consider (for example χ_A functions) have no such components. Therefore, we rather use the $SBV(\Omega)$ space (see [2] for example) which is the space of functions in $BV(\Omega)$ whose derivative has no singular Cantor component. The functions of $SBV(\Omega)$ have two components : one is regular and is defined almost everywhere on Ω (for the Lebesgue measure).

The support S of the second one generally satisfies $\mathcal{H}^{N-1}(S) \neq 0$.
The problems we finally consider writes

$$\mathrm{Min}\ \left\{\frac{1}{2}\int_\Omega (p-g)^2\,dx + \beta\mathcal{H}^{N-1}(S_p) : p \in SBV(\Omega), p \in \{0,1\} \text{ a.e.}\right\}. \tag{$\mathcal{P}$}$$

2.2. **Approximate model.** The study of the Mumford-Shah model is still challenging and it is easier to consider approximate versions. Modica and Mortola ([19]) prove a Γ-convergence result for functional

$$F_\varepsilon(u) = \int_\Omega \left(\varepsilon|\nabla u|^2 + \frac{W(u)}{\varepsilon}\right)\,dx \tag{2.4}$$

to the area functional for surface of dimension $N-1$, where W is a double-well potential. Inspired by this work, we set

$$\mathcal{E}_\varepsilon(p) = \frac{1}{2}\int_\Omega (p-g)^2\,dx + \beta\int_\Omega \left(9\varepsilon|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon}\right)\,dx \tag{2.5}$$

and define the approximate problem as

$$\min\{\mathcal{E}_\varepsilon(p)\,|\,p \in H^1(\Omega)\}. \tag{$\mathcal{P}_\varepsilon$}$$

The choice of coefficients $\lambda_1 = 9\varepsilon$ and $\lambda_2 = \frac{1}{\varepsilon}$ in this model may seem surprising. In fact, it is required that these parameters verify $2\sqrt{\lambda_1\lambda_2} = 6$ to get a Γ-convergence result for the approximate model (see [25]). It is easy to prove that ($\mathcal{P}_\varepsilon$) has at least an optimal solution p_ε. However, as $\mathcal{E}_\varepsilon$ is not convex, we get no uniqueness.

Problem ($\mathcal{P}_\varepsilon$) is a suitable approximation of ($\mathcal{P}$). Indeed we have the following convergence result:

Theorem 2.1 ([25]). *For every $\varepsilon > 0$, let p_ε be a solution to $(\mathcal{P}_\varepsilon)$. Then we may extract a subsequence p_{ε_n} that converges a.e. to a binary function $\bar{p}$ ($\bar{p}(x) \in \{0,1\}$a.e. x) which is a solution to $(\mathcal{P})$.*

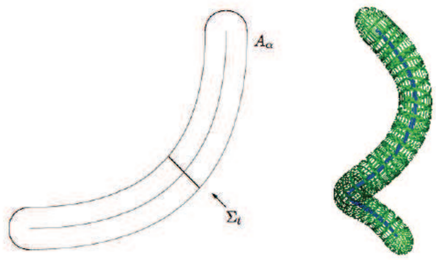
We refer to [5] for definition of the Γ-convergence and related properties.

3. Tube detection

The functional $\mathcal{E}_\varepsilon$ is not convex because of the term $p \to \displaystyle\int_\Omega \frac{p^2(1-p)^2}{\varepsilon}\,dx$. Therefore we cannot ensure the uniqueness of the solution to $(\mathcal{P}_\varepsilon)$. As we get existence however, we must refine the model adding a geometrical prior. So we consider the minimization of $\mathcal{E}_\varepsilon$ on the set of *tubes* (that we are going to define). This will provide a unique minimizer according to the tube geometry.

3.1. Modeling a tube.

We present here a description of what we call (thin) *tubes* both for the 2D and 3D dimension. Roughly speaking, we define a tube as a symmetric object of codimension 1 whose length ℓ is much greater that the width α. Let Γ be a parametrized curve in Ω : we get the tube by *thickening* the curve to get a symmetric object of width $\alpha > 0$.



(a) Dimension 2 (b) Dimension 3

Figure 3.1. Tubes A_α

Let us detail the 3D representation of such tubes. The 2D case is straightforward (deleting one dimension).

3.1.1. *The parametrized curve* Γ. Let $\Gamma \subset \Omega$ be a $\mathcal{C}^2$ curve in $\Omega \subset \mathbb{R}^3$. We use a parametrization with a curvilinear abscissa $F\colon [0,\ell] \to \Gamma$ and assume the following regularity condition :

$$(\mathcal{H}_\Gamma)\ \begin{cases} F \text{ is bijective,} \\ F \text{ is } \mathcal{C}^2 \text{ and } \forall t \in [0,\ell],\ |F'(t)| = 1, \\ F \text{ is biregular : } \forall t \in [0,\ell],\ \dim \operatorname{Span}(F^{'}(t), F^{''}(t)) = 2, \end{cases}$$

where $F'(t)$ is the first derivative, $F^{''}(t)$ the second derivative. Assumption $(\mathcal{H}_\Gamma)$ allows to define a Frenet–Serret frame whose main properties are recalled thereafter:

Proposition 3.1. *Let $\mathbb{S}^2$ be the unit sphere of $\mathbb{R}^3$ and assume $(\mathcal{H}_\Gamma)$is fulfilled, then there exist*

- *$T\colon [0,\ell] \to \mathbb{S}^2$ the unit vector tangent to the curve, pointing in the direction of motion,*
- *$N\colon [0,\ell] \to \mathbb{S}^2$ the normal unit vector, the derivative of T with respect to the arclength parameter of the curve, divided by its length.,*

- $B\colon [0,\ell\,] \to \mathbb{S}^2$ *the binormal unit vector, the cross product of T and N.*

Moreover

$$\frac{d}{dt}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{3.1}$$

Functions γ (curvature) and τ (torsion) are scalar functions and γ is nonnegative. The curvature γ is the curvature radius inverse.

Remark 3.1. *In the 2D case the Frenet–Serret frame reduces to T and N. Existence conditions are the same and the differential characterization of the frame is*

$$\frac{d}{dt}\begin{pmatrix} T \\ N \end{pmatrix} = \begin{pmatrix} 0 & \gamma \\ -\gamma & 0 \end{pmatrix}\begin{pmatrix} T \\ N \end{pmatrix}. \tag{3.2}$$

Eventually, we have to set an additional hypothesis to get a local parametrization in the neighborhood of Γ : we need the curvature radius to be large enough. If it was smaller than the diameter of the tube, this would correspond to the case where the tube fall back on itself. Therefore, we assume

$$\forall t \in [0,\ell], \quad \alpha < \inf\left\{\frac{1}{\gamma(t)}\right\}. \tag{3.3}$$

Remark 3.2. *It is sufficient to assume there exists $\rho > 0$ such that*

$$\forall t \in [0,\ell], \quad \frac{\alpha}{2} + \rho < \inf \frac{1}{\gamma(t)}\ .$$

We chose $\rho = \frac{\alpha}{2}$ for the sake of simplicity

We may now define the tube A_α with thickness α around Γ as

$$A_\alpha = \{x \in \Omega\colon \mathbf{d}(x,\Gamma) < \alpha/2\} \text{ and } g = \chi_{A_\alpha} = \begin{cases} 1 & \text{on } A_\alpha \\ 0 & \text{elsewhere.} \end{cases} \tag{3.4}$$

Here $\mathbf{d}$ is the euclidean distance in $\mathbb{R}^N$. Let us divide A_α into three sub-areas: the two ends B^0_α, B^ℓ_α and the body C_α. More precisely

$$\begin{aligned} B^0_\alpha &= \{x \in A_\alpha\colon \|x - F(0)\| = \mathbf{d}(x,\Gamma)\}, \\ B^\ell_\alpha &= \{x \in A_\alpha\colon \|x - F(\ell)\| = \mathbf{d}(x,\Gamma)\} \end{aligned} \tag{3.5}$$

and $C_\alpha = A_\alpha \setminus (B^0_\alpha \cup B^\ell_\alpha)$.

3.1.2. *Parametrization of the tube.* In order to perform calculations, we must specify the tube parametrization. For this, we consider the ends and the body separately and use spherical coordinates (or polar coordinates for the 2D case).

Proposition 3.2. *Assume $(\mathcal{H}_\Gamma)$ and (3.3). Then we may define*

- *2D case ($N = 2$) :*

$$\begin{aligned} \Phi_C\colon\ [0,\ell] \times \left]-\frac{\alpha}{2},\frac{\alpha}{2}\right[&\to C_\alpha \\ (t,r) &\mapsto F(t) + rN(t), \end{aligned}$$

$$\begin{aligned} \Phi_{B^0}\colon \left]0,\frac{\alpha}{2}\right[\times \left]0,\pi\right[&\to B^0_\alpha \\ (r,\theta) &\mapsto F(0) + r\cos(\theta)N(0) - r\sin(\theta)T(0), \end{aligned}$$

$$\begin{aligned} \Phi_{B^\ell}\colon \left]0,\frac{\alpha}{2}\right[\times \left]0,\pi\right[&\to B^\ell_\alpha \\ (r,\theta) &\mapsto F(\ell) + r\cos(\theta)N(\ell) + r\sin(\theta)T(\ell); \end{aligned}$$

- *3D case ($N = 3$) :*

$$\begin{array}{rcl} \Phi_C & : & [0,\ell] \times \left]-\frac{\alpha}{2},\frac{\alpha}{2}\right[\times \left]-\frac{\pi}{2},\frac{\pi}{2}\right[\to C_\alpha \\ (t,r,\theta) & \mapsto & F(t) + r\cos(\theta)N(t) + r\sin(\theta)B(t), \end{array}$$

$$\begin{array}{rcl} \Phi_{B^0} & : & \left]0,\frac{\alpha}{2}\right[\times]0,2\pi[\times \left]0,\frac{\pi}{2}\right[\to B^0_\alpha \\ (r,\theta,\phi) & \mapsto & F(0) + r\cos(\phi)(\cos(\theta)N(0) + \sin(\theta)B(0)) - r\sin(\phi)T(0), \end{array}$$

$$\begin{array}{rcl} \Phi_{B^\ell} & : & \left]0,\frac{\alpha}{2}\right[\times]0,2\pi[\times \left]0,\frac{\pi}{2}\right[\to B^\ell_\alpha \\ (r,\theta,\phi) & \mapsto & F(\ell) + r\cos(\phi)(\cos(\theta)N(\ell) + \sin(\theta)B(\ell)) + r\sin(\phi)T(\ell). \end{array}$$

Moreover, Φ_C is a local diffeomorphism whose jacobian is

$$\begin{array}{rcll} J\Phi_C(t,r) = & 1 - r\gamma(t) & \text{if } N = 2, \\ J\Phi_C(t,r,\theta) = & r(1 - r\cos(\theta)\gamma(t)) & \text{if } N = 3. \end{array}$$

Proof. Using the 3D Frenet–Serret formulas (3.1) gives

$$J\Phi_C(t,r,\theta) = \begin{vmatrix} 1-\gamma(t)r\cos\theta & 0 & 0 \\ -\tau(t)r\sin\theta & \cos\theta & -r\sin\theta \\ \tau(t)r\cos\theta & \sin\theta & r\cos\theta \end{vmatrix} = r(1 - r\cos(\theta)\gamma(t)).$$

As $r < \alpha/2$ and the curvature radius of Γ is always greater than $\alpha/2$ so that

$$\forall t \in [0,\ell], \quad |1 - r\cos(\theta)\gamma(t))| \neq 0.$$

This proves that Φ_C is a local diffeomorphism. □

Now, we gather the respective parametrizations Φ_{B^0}, Φ_{B^ℓ} and Φ_C in an atlas of A_α.

Definition 3.1. *Assume $(\mathcal{H}_\Gamma)$ and (3.3). We say that (A_α, Γ) is a tube if the family $\{\Phi_{B^0}, \Phi_{B^\ell}, \Phi_C\}$ is a $\mathcal{C}^1$-atlas of A_α.*

3.1.3. *The set $\mathcal{F}_\alpha$ of tubes of width α.* From now we assume that (Γ, A_α) is a tube (as in definition 3.1). We now define a set of feasible functions to describe such a tube.

Definition 3.2. *Let (Γ, A_α) be a tube satisfying $(\mathcal{H}_\Gamma)$ and (3.3). The space $\mathcal{F}_\alpha \subset H^1_0(\Omega)$ is defined as the space of $H^1_0(\Omega)$ functions p such that*

a) *for almost every $x \in \Omega \setminus A_\alpha$, $p(x) = 0$;*
b) *for almost every $(x, \tilde{x}) \in A_\alpha \times A_\alpha$,*

$$\mathbf{d}(x,\Gamma) = \mathbf{d}(\tilde{x},\Gamma) \quad \Rightarrow \quad p(x) = p(\tilde{x}).$$

We end with an obvious property of $\mathcal{F}_\alpha$ functions.

Lemma 3.1. *Let (Γ, A_α) be a tube satisfying $(\mathcal{H}_\Gamma)$ and (3.3). Then, for almost every $(t,\tilde{t}) \in [0,\ell]^2$, $(\theta,\tilde{\theta}) \in (\left]-\frac{\pi}{2},\frac{\pi}{2}\right[)^2$ and $(r,\tilde{r}) \in \left(\left]-\frac{\alpha}{2},\frac{\alpha}{2}\right[\right)^2$, we get:*

$$|r| = |\tilde{r}| \quad \Rightarrow \quad p(\Phi(t,r,\theta)) = p(\Phi(\tilde{t},\tilde{r},\tilde{\theta})).$$

3.1.4. *Minimization problem with tube constraints.* If we consider all the tubes that satisfy assumption $(\mathcal{H}_\Gamma)$, the minimizing problem is

$$\min_{\Gamma\in\mathcal{C}^2}\min_{p\in\mathcal{F}_\alpha}\mathcal{E}_{\varepsilon(p)}.$$

For this formulation, difficulties arise from the fact that self intersections of Γ are possible and then A_α is not a tube. So, we assume that Γ is known and we focus on the *low-level* problem

$$\min_{p\in\mathcal{F}_\alpha}\mathcal{E}_\varepsilon(p). \tag{$\mathcal{P}_{\varepsilon,\alpha}$}$$

Here

$$\mathcal{E}_\varepsilon(p)=\frac{1}{2}\int_{A_\alpha}(p-1)^2dx+\beta\int_{A_\alpha}\left(9\varepsilon|\nabla p|^2+\frac{p^2(p-1)^2}{\varepsilon}\right)dx \tag{3.6}$$

since $g:=\chi_{A_\alpha}$ and functions in $\mathcal{F}_\alpha$ have their support in A_α. The end of the paper is devoted to existence and uniqueness result for $\mathcal{P}_{\varepsilon,\alpha}$. In addition, qualitative properties of the solution will provide parameters tuning with respect to the width α and the length ℓ.

3.2. **Solving the problem $(\mathcal{P}_{\varepsilon,\alpha})$.** We first give an existence result. Then, under an hypothesis of smallness for β, we prove uniqueness of the solution.

Theorem 3.1 (Existence)**.** *Problem $(\mathcal{P}_{\varepsilon,\alpha})$ has at least an optimal solution.*

Proof. Let $(p_n)_n$ be a minimizing sequence. As $(p_n)_n$ and $(\nabla p_n)_n$ are bounded in $L^2(\Omega)$, the sequence $(p_n)_n$ is bounded in $H^1_0(\Omega)$. Moreover $H^1_0(\Omega)$ is compactly embedded in $L^4(\Omega)$ ($N\leq 3$, see [23]). Therefore, one may extract a subsequence (denoted similarly) that weakly converges to $\bar p$ in $H^1_0(\Omega)$ and strongly in $L^4(\Omega)$. The lower semi -continuity of $\mathcal{E}_\varepsilon$ then gives

$$\mathcal{E}_\varepsilon(\bar p)\leq\liminf_{n\to+\infty}\mathcal{E}_\varepsilon(p_n).$$

It remains to prove that $\bar p\in\mathcal{F}_\alpha$. As $H^1_0(\Omega)$ is compactly embedded in $L^1(\Omega)$ the sequence $(p_n)_n$ converges to $\bar p$ almost everywhere (up to a subsequence). The symmetry properties of definition 3.2 are kept by taking the limit. This gives $\bar p\in\mathcal{F}_\alpha$. $\square$

In the sequel we set

$$\forall t\in\mathbb{R},\quad F_{\beta,\varepsilon}(t)=\frac{1}{2}(t-1)^2+\frac{\beta}{\varepsilon}(t^2-t)^2 \tag{3.7}$$

and

$$\forall t\in\mathbb{R},\quad f_{\beta,\varepsilon}(t)=\frac{1}{2}F'_{\beta,\varepsilon}(t)=\frac{\beta}{\varepsilon}(2t^3-3t^2)+(\frac{1}{2}+\frac{\beta}{\varepsilon})t-\frac{1}{2}\ . \tag{3.8}$$

Theorem 3.2 (Optimality condition)**.** *Let $\bar p$ be a solution to $(\mathcal{P}_{\varepsilon,\alpha})$. Then $\bar p\in\mathcal{F}_\alpha$ satisfies*

$$\forall\varphi\in\mathcal{F}_\alpha,\quad\int_{A_\alpha}(9\beta\varepsilon\nabla\bar p(x)\nabla\varphi(x)+f_{\beta,\varepsilon}(p(x))\varphi(x))\,dx=0\ . \tag{3.9}$$

Proof. A classical computation gives

$$\begin{aligned}\forall\varphi\in\mathcal{F}_\alpha,\quad <\nabla\mathcal{E}_\varepsilon(\bar p),\varphi> &=\int_{A_\alpha}\big(18\beta\varepsilon\nabla\bar p(x)\nabla\varphi(x)+F'_{\beta,\varepsilon}(\bar p(x))\varphi(x)\big)\,dx,\\ &=2\int_{A_\alpha}(9\beta\varepsilon\nabla\bar p(x)\nabla\varphi(x)+f_{\beta,\varepsilon}(\bar p(x))\varphi(x))\,dx.\end{aligned}$$

Every solution $\bar p\in\mathcal{F}_\alpha$ to $(\mathcal{P}_{\varepsilon,\alpha})$ satisfies

$$\forall\varphi\in\mathcal{F}_\alpha,\quad <\nabla\mathcal{E}_\varepsilon,\varphi-\bar p>\geq 0\ ,$$

that is (since $\mathcal{F}_\alpha$ is a linear space)

$$\forall\varphi\in\mathcal{F}_\alpha,\quad <\nabla\mathcal{E}_\varepsilon(\bar p),\varphi>=0\ .$$

□

Theorem 3.3 (Uniqueness). *If $\beta \le \varepsilon$, then problem $(\mathcal{P}_{\varepsilon,\alpha})$ has a unique solution.*

Proof. Let p_1 and p_2 be two solutions of $(\mathcal{P}_{\varepsilon,\alpha})$. As p_1 and p_2 belong to $\mathcal{F}_\alpha$ one may choose $\varphi = p_1 - p_2$ in (3.9): using this equality with p_1 and p_2 respectively and subtracting gives

$$\int_{A_\alpha} 9\beta\varepsilon|\nabla(p_1 - p_2)|^2 + (f_{\beta,\varepsilon}(p_1) - f_{\beta,\varepsilon}(p_2))(p_1 - p_2)\, dx = 0 \ .$$

As $(f_{\beta,\varepsilon}(p_1) - f_{\beta,\varepsilon}(p_2))(p_1 - p_2) = f'_{\beta,\varepsilon}(p_1 + \theta p_2)(p_1 - p_2)^2$ with $\theta \in [0,1]$, it is sufficient that $f'_{\beta,\varepsilon} \ge 0$ to get $p_1 = p_2$. As $f'_{\beta,\varepsilon}(t) = \frac{6\beta}{\varepsilon}t^2 - \frac{6\beta}{\varepsilon}t + \left(\frac{1}{2} + \frac{\beta}{\varepsilon}\right)$, then $f'_{\beta,\varepsilon} \ge 0$ for $\beta \le \varepsilon$. □

Remark 3.3. *The hypothesis $\beta \le \varepsilon$ is not restrictive in practice. For numerics, ε corresponds to the distance between the sets $\{x \in \Omega\colon p(x) \approx 0\}$ and $\{x \in \Omega\colon p(x) \approx 1\}$. For the need of stability, ε has to be chosen greater than the thickness of the tube (see [11]). On the other hand, β is the regularization coefficient. For detection of thin structures, this coefficient has to be small. Thus, the condition $\beta \le \varepsilon$ is easily fulfilled in practice.*

The first significant result of this section is the existence of a unique solution providing $\beta \le \varepsilon$. More informations come from the optimality conditions that we make precise now. Indeed, the constraint $p \in \mathcal{F}_\alpha$ does not go directly to a partial differential equation from (3.9): we cannot ensure that the solution of such an equation (to be computed in the dual of $\mathcal{F}_\alpha$) exists and belongs to $\mathcal{F}_\alpha$. In addition, the numerical description of $\mathcal{F}_\alpha$ is difficult. For all these reasons, we first show that the 2D/3D problem $(\mathcal{P}_{\varepsilon,\alpha})$ can be reduced to a problem in **one dimension**. We can then give specific properties of the solution and provide an automatic selection of parameters β and ε with respect to α and ℓ.

3.3. **Reduction to a one dimensional problem.** To reduce the problem $(\mathcal{P}_{\varepsilon,\alpha})$ to a 1D problem, we exhibit a diffeomorphism that allows to fully describe a tube (through its parametrization) with a single variable. This is made possible by the very specific definition of the concept of tube. We will have to relax some assumptions later to handle the case of more general tubes.

The 1D problem we obtain is formulated in a weighted Sobolev space where the weight ω is related to the geometry of the tube and (therefore) the space dimension. The case of dimensions 2 and 3 are treated in the same way with a significant difference in 3D since the weight ω vanishes at 0.

Assume that (Γ, A_α) is a tube as in the definition 3.1. The purpose of this section is to obtain an expression of the energy when restricted to $\mathcal{F}_\alpha$. In what follows we set

$$\omega(r) = \frac{\mathcal{H}^{N-1}(\partial A_r)}{2} \ ,$$

where $N = 2, 3$ is the space dimension, $\mathcal{H}$ the Hausdorff measure and ∂A_r the boundary of the tube $A_{|r|}$ (with length ℓ). A quick computation gives

$$\omega(r) = \begin{cases} \ell + \pi|r| & \text{if } N = 2 \ , \\ \pi\ell|r| + 2\pi r^2 & \text{if } N = 3 \ . \end{cases} \tag{3.10}$$

Definition 3.3. *Let be $I_\alpha = [-\frac{\alpha}{2}, \frac{\alpha}{2}]$. The weighted Sobolev space $H^1_\omega(I_\alpha)$ is defined as*

$$H^1_\omega(I_\alpha) := \{q \in L^2(I_\alpha) \mid \int_{I_\alpha} \left(|q|^2 + |q'|^2\right) \omega(r)dr < +\infty\},$$

where ω is given by (3.10). This space is endowed with the norm

$$\|q\|^2_{H^1_\omega(I_\alpha)} = \int_{I_\alpha} \left(|q|^2 + |q'|^2\right) \omega(r)dr.$$

It is easy to see that $H^1(I_\alpha)$ is continuously embedded in $H^1_\omega(I_\alpha)$ since ω is bounded on I_α. The converse embedding is true if $N = 2$.

Lemma 3.2. *If $N = 2$, then $H^1_\omega(I_\alpha) = H^1(I_\alpha)$ and the associated norms are equivalent. If $N = 3$, then $H^1_\omega(I_\alpha) \subset \mathcal{C}^0(I_\alpha \setminus \{0\})$.*

Proof. The proof is given in appendix. □

If $N = 2$, $H^1_\omega(I_\alpha) = H^1(I_\alpha) \subset \mathcal{C}^0(I_\alpha)$ (continuous functions on I_α) and the corresponding functions are defined everywhere. In particular the trace on the boundary makes sense. In the 3D case the result is different since $\omega(0) = 0$. Therefore functions in $H^1_\omega(I_\alpha)$ may have a singularity at 0. However, we still have a continuity results *outside* 0.
We may define 1D spaces analogous to $H^1_0(\Omega)$ and $\mathcal{F}_\alpha$:

Definition 3.4. *Let ω be defined by (3.10). The space $H^1_{\omega,0}(I_\alpha)$ is the space of $H^1_\omega(I_\alpha)$ functions that vanish at $-\frac{\alpha}{2}$ and $\frac{\alpha}{2}$. The space $\mathcal{G}^\omega_\alpha$ is the space of even functions of $H^1_{\omega,0}(I_\alpha)$.*

The correspondence between 2D/3D case and 1D case is described in next proposition:

Proposition 3.3. *The following application Θ is an isomorphism from $\mathcal{F}_\alpha$ to $\mathcal{G}^\omega_\alpha$:*

$$\begin{array}{rcl} \Theta : \mathcal{F}_\alpha & \to & \mathcal{G}^\omega_\alpha \\ p & \to & \left\{ \begin{array}{rcl} q : I_\alpha & \to & \mathbb{R} \\ r & \mapsto & p(\Phi_C(0,r)), \end{array} \right. \end{array}$$

where Φ_C is defined with (3.2). Moreover, if $q = \Theta(p)$ then

$$\|p\|_{H^1(\Omega)} = \|q\|_{H^1_\omega(I_\alpha)}. \tag{3.11}$$

Proof. The proof is given in appendix.

□

Remark 3.4. *Let Σ_t be the t- slice of A_α :*

$$\Sigma_t = \left\{ \begin{array}{ll} \{\Phi_C(t,r) \colon r \in I_\alpha\} & \textit{if } N = 2, \\ \{\Phi_C(t,r,\theta) \colon (r,\theta) \in I_\alpha \times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\} & \textit{if } N = 3. \end{array} \right.$$

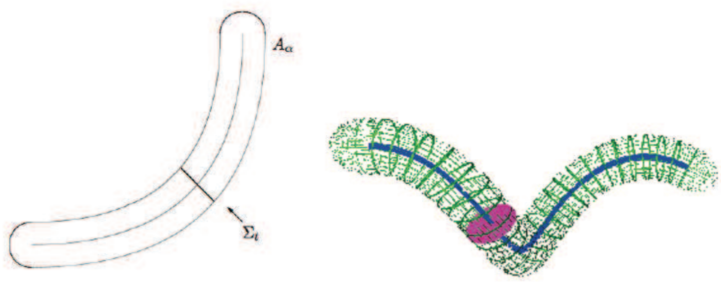


FIGURE 3.2. Slice Σ_t for $N = 2, 3$

The bijectivity of Θ means that any element of $\mathcal{F}_\alpha$ is characterized by its image in the slice Σ_0 or any other slice Σ_t of A_α. This comes from the (strong) assumption we made on the geometry of the tube whose width α is constant. This assumption will be relaxed in the future to consider tubes with varying width.

Now we can perform the change of variables $q = \Theta(p)$ that provides an equivalent 1D formulation of problem $(\mathcal{P}_{\varepsilon,\alpha})$. We define $G_\varepsilon:\ \mathcal{G}^\omega_\alpha \to \mathbb{R}^+$ as follows

$$\forall q \in \mathcal{G}^\omega_\alpha, \quad G_\varepsilon(q) = \mathcal{E}_\varepsilon(\Theta^{-1}(q)) \ .$$

Let us give an explicit expression of G_ε:

Proposition 3.4. *The function $G_\varepsilon : \mathcal{G}^\omega_\alpha \to \mathbb{R}^+$ satisfies*

$$\forall q \in \mathcal{G}^\omega_\alpha, \quad G_\varepsilon(q) = \int_{I_\alpha} \left[9\varepsilon\beta|q'|^2 + \frac{1}{2}(1-q)^2 + \frac{(q-q^2)^2}{\varepsilon}\beta \right] \omega(r)\, dr \tag{3.12}$$

where ω has been defined in (3.10).

Proof. The proof is given in appendix. □

We call next **reduced** problem on $\mathcal{G}^\omega_\alpha$ the following

$$\min_{q\in\mathcal{G}^\omega_\alpha} G_\varepsilon(q). \tag{$\mathcal{P}^\omega_{\varepsilon,\alpha}$}$$

We just proved that we may reduce $(\mathcal{P}_{\varepsilon,\alpha})$ to a 1D problem. More precisely:

Theorem 3.4. *Assume that $(\mathcal{H}_\Gamma)$ and (3.3) are fulfilled.*
The function p is solution to $(\mathcal{P}_{\varepsilon,\alpha})$ if and only if $\Theta(p)$ is solution to $(\mathcal{P}^\omega_{\varepsilon,\alpha})$ where

- *Θ is given by*

$$\begin{array}{rcl} \Theta : \mathcal{F}_\alpha & \to & \mathcal{G}^\omega_\alpha \\ p & \mapsto & \left\{ \begin{array}{rcl} q : I_\alpha & \to & \mathbb{R} \\ r & \mapsto & p(\Phi_C(0,r)), \end{array}\right. \end{array}$$

- *$\mathcal{F}_\alpha$ is given by definition 3.2 and $\mathcal{G}^\omega_\alpha$ by definition 3.4*
- *G_ε is given by (3.12) and $H^1_\omega(I_\alpha)$ by (3.3) with*
  - *2D case : $\omega(r) = \ell + \pi|r|$ and*

$$\begin{array}{rcl} \Phi_C\colon\ [0,\ell] \times \left]-\frac{\alpha}{2}, \frac{\alpha}{2}\right[& \to & C_\alpha \\ (t,r) & \mapsto & F(t) + rN(t), \end{array}$$

  - *3D case : $\omega(r) = \pi\ell r + 2\pi r^2$ and*

$$\begin{array}{rcl} \Phi_C & : & [0,\ell] \times \left]-\frac{\alpha}{2}, \frac{\alpha}{2}\right[\times \left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\to C_\alpha \\ (t,r,\theta)) & \mapsto & F(t) + r\cos(\theta)N(t) + r\sin(\theta)B(t). \end{array}$$

3.4. **Solution properties.** Theorem 3.4 is the key result of this paper: indeed we may now obtain quantitative and qualitative properties of the solution to $(\mathcal{P}_{\varepsilon,\alpha})$ from the solution to $(\mathcal{P}^\omega_{\varepsilon,\alpha})$.

With the symmetry properties of functions in $\mathcal{G}^\omega_\alpha$ it is easy to check that the restriction of the solution $\bar{q}$ of $(\mathcal{P}^\omega_{\varepsilon,\alpha})$ (with $\beta \le \varepsilon$) to $(0, \frac{\alpha}{2})$ is the unique solution of

$$\min_{g\in\mathcal{G}^{\omega,+}_\alpha} G^+_\varepsilon(g) \tag{3.13}$$

, where

$$\mathcal{G}^{\omega,+}_\alpha = \left\{ q_{|]0,\frac{\alpha}{2}]} |\ q \in \mathcal{G}^\omega_\alpha \right\}$$

and

$$G^+_\varepsilon(g) = \int_0^{\frac{\alpha}{2}} \left(9\beta\varepsilon|g'|^2 + \frac{1}{2}(1-g)^2 + \beta\frac{(g-g^2)^2}{\varepsilon} \right) \omega(r)\, dr \ = \int_0^{\frac{\alpha}{2}} \left(9\beta\varepsilon|g'|^2 + F_{\beta,\varepsilon}(g)\right) \omega(r)\, dr,$$

with the notations introduced in (3.7).

Theorem 3.5. *Assume $\beta \leq \varepsilon$. Let $\bar p$ be the unique solution to $(\mathcal{P}_{\varepsilon,\alpha})$ and $\bar q = \Theta(\bar p)$. Then $\bar q$ (and $\bar p$) takes its values in $[0,1]$. In particular $\bar q \in L^\infty(I_\alpha)$.*

Proof. It is sufficient to prove that

$$\forall r \in]0, \frac{\alpha}{2}], \quad 0 \leq \bar q(r) \leq 1 \ .$$

Lemma 3.2 ensures that $\bar q$ is continuous on $]0,\frac{\alpha}{2}]$ in the 3D case and continuous on $[0,\frac{\alpha}{2}]$ in the 2D case. So if $N=2$, we get $0 \leq \bar q(0) \leq 1$ by continuity.

• Let us prove first that

$$\forall r \in]0, \frac{\alpha}{2}], \quad \bar q(r) \leq 1.$$

Define $\varphi = \min(\bar q, 1)$ on $]0,\frac{\alpha}{2}]$ so that $\varphi \in \mathcal{G}_\alpha^{\omega,+}$. Then

$$\int_0^{\frac{\alpha}{2}} 9\beta\varepsilon|\varphi'|^2\omega(r)\,dr \leq \int_0^{\frac{\alpha}{2}} 9\beta\varepsilon|\bar q'|^2\omega(r)\,dr \ .$$

We recall that $F'_{\beta,\varepsilon}(t) = 2f_{\beta,\varepsilon}(t)$, where $f_{\beta,\varepsilon}$ is given by (3.8). We have proved in theorem 3.3 that $\beta \leq \varepsilon$ implies $f'_{\beta,\varepsilon} \geq 0$. Therefore, $F''_{\beta,\varepsilon} > 0$ and $F'_{\beta,\varepsilon}$ is increasing. As $F'_{\beta,\varepsilon}(1) = 0$, the function $F'_{\beta,\varepsilon}$ is negative on $]-\infty,1]$ and nonnegative on $[1,+\infty[$. Then the function $F_{\beta,\varepsilon}$ is decreasing $]-\infty,1]$ and increasing on $[1,+\infty[$, so that

$$\begin{array}{ll} F_{\beta,\varepsilon}(\varphi(r)) = F_{\beta,\varepsilon}(\bar q(r)) & \text{if } \bar q(r)) \leq 1 \ , \\ F_{\beta,\varepsilon}(\varphi(r)) \leq F_{\beta,\varepsilon}(\bar q(r)) & \text{if } \bar q(r)) \geq 1 = \varphi(r) \ . \end{array}$$

As $\omega \geq 0$ it comes $G_\varepsilon^+(\varphi) \leq G_\varepsilon^+(\bar q)$ and with the uniqueness of the solution, this yields $\varphi = \bar q$, so that $\bar q \leq 1$.

• We prove similarly that

$$\forall r \in]0, \frac{\alpha}{2}], \quad q(r) \geq 0.$$

Set $\varphi = \max(\bar q, 0)$ on $]0,\frac{\alpha}{2}]$ so that $\varphi \in \mathcal{G}_\alpha^{\omega,+}$ and

$$\int_0^{\frac{\alpha}{2}} 9\beta\varepsilon|\varphi'|^2\omega(r)\,dr \leq \int_0^{\frac{\alpha}{2}} 9\beta\varepsilon|\bar q'|^2\omega(r)\,dr \ .$$

Furthermore

$$\begin{array}{ll} F_{\beta,\varepsilon}(\varphi(r)) = F_{\beta,\varepsilon}(\bar q(r)) & \text{if } \bar q(r)) \geq 0 \ , \\ 1 = F_{\beta,\varepsilon}(0) \leq F_{\beta,\varepsilon}(\bar q(r)) & \text{if } \bar q(r)) \leq 0 \ , \end{array}$$

since $F_{\beta,\varepsilon}$ is decreasing on $]-\infty,0]$. As $\omega \geq 0$, we get

$$G_\varepsilon^+(\varphi) \leq G_\varepsilon^+(\bar q) + \int_{\bar q \leq 0} (1 - F(\bar q))\omega(r)\,dr \leq G_\varepsilon^+(\bar q) \ .$$

As before, $\varphi = \bar q$ and $\bar q \geq 0$. □

Now, we make the optimality condition precise : let $\bar p$ be the unique solution to $(\mathcal{P}_{\varepsilon,\alpha})$ and $\bar q = \Theta(\bar p)$. Then

$$\forall \psi \in \mathcal{G}_\alpha^\omega, \quad \int_{I_\alpha} \left(9\beta\varepsilon\bar q'\psi' + f_{\beta,\varepsilon}(q)\psi\right)\omega(r)\,dr = 0 \ . \tag{3.14}$$

Let us denote

$$\mathcal{H}^1_\omega(0,\frac{\alpha}{2}) := \left\{\varphi \in H^1_\omega(0,\frac{\alpha}{2}),\ \varphi(\frac{\alpha}{2}) = 0\right\},$$

where ω is defined with (3.10). It is a linear subspace of $\mathcal{C}^0([0,\frac{\alpha}{2}])$ for $N=2$ and $\mathcal{C}^0(]0,\frac{\alpha}{2}])$ for $N=3$. For every function $\varphi \in \mathcal{H}^1_\omega(0,\frac{\alpha}{2})$ we set

$$\psi(x) = \left\{ \begin{array}{ll} \varphi(x) & \text{if } x > 0, \\ \varphi(-x) & \text{if } x < 0. \end{array}\right.$$

The function ψ belongs to $\mathcal{G}^\omega_\alpha$ and with (3.14)

$$\int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left(9\beta\varepsilon\bar{q}'\psi' + f_{\beta,\varepsilon}(\bar{q})\psi\right)\omega(r)\,dr = 2\int_0^{\frac{\alpha}{2}} \left(9\beta\varepsilon\bar{q}'\psi' + f_{\beta,\varepsilon}(\bar{q})\psi\right)\omega(r)\,dr$$

$$= 2\int_0^{\frac{\alpha}{2}} \left(9\beta\varepsilon\bar{q}'\varphi' + f_{\beta,\varepsilon}(\bar{q})\varphi\right)\omega(r)\,dr = 0\ .$$

Finally

$$\forall\varphi\in\mathcal{H}^1_\omega(0,\frac{\alpha}{2}),\quad \int_0^{\frac{\alpha}{2}} \left(9\beta\varepsilon\bar{q}'(r)\varphi'(r) + f_{\beta,\varepsilon}(\bar{q}(r))\varphi(r)\right)\omega(r)\,dr = 0\ .$$

Choose $\varphi\in\mathcal{D}(0,\frac{\alpha}{2})$ and integrate by parts gives

$$-9\beta\varepsilon\left(\omega\bar{q}'\right)' + \omega f_{\beta,\varepsilon}(\bar{q}) = 0 \text{ in } (0,\frac{\alpha}{2})\ , \tag{3.15}$$

in the sense of distributions and in addition $\bar{q}(\frac{\alpha}{2}) = 0$.
Now, choose $\varphi\in\mathcal{C}^1(0,\frac{\alpha}{2})$ such that $\varphi(\frac{\alpha}{2}) = 0$ and $\varphi(0)\neq 0$. An integration by parts gives

$$\int_0^{\frac{\alpha}{2}} \bar{q}'\varphi'\omega(r)\,dr = -\int_0^{\frac{\alpha}{2}} (\omega\bar{q}')'\varphi\,dr - \bar{q}'(0)\omega(0)\varphi(0)\ ,$$

and with (3.15) we obtain $\bar{q}'(0)\omega(0)\varphi(0) = 0$, that is $\bar{q}'(0)\omega(0) = 0$. Consequently, if $N = 2$ ($\omega(0) = \ell\neq 0$) we get $q'(0) = 0$ and we may describe the 2D solution.

3.5. **2D case.**

Theorem 3.6 (Euler equation). *Assume $\beta\leq\varepsilon$. Let $\bar{p}$ be the unique solution to $(\mathcal{P}_{\varepsilon,\alpha})$ and $\bar{q} = \Theta(\bar{p})$. Then $\bar{q}\in\mathcal{G}^\omega_\alpha$ is solution to the boundary problem*

$$\left\{\begin{array}{l} -9\beta\varepsilon\,(\omega\bar{q}')' + \omega f_{\beta,\varepsilon}(\bar{q}) = 0 \text{ in } (0,\frac{\alpha}{2})\ ,\\ \bar{q}(\frac{\alpha}{2}) = 0, \bar{q}'(0) = 0.\end{array}\right. \tag{3.16}$$

In that case $\bar{q}\in\mathcal{C}^2(I_\alpha)$ and is the (strong) solution to

$$\left\{\begin{array}{l} -9\beta\varepsilon\,(\omega\bar{q}')' + \omega f_{\beta,\varepsilon}(\bar{q}) = 0 \text{ in } I_\alpha\ ,\\ \bar{q}(-\frac{\alpha}{2}) = \bar{q}(\frac{\alpha}{2}) = 0.\end{array}\right. \tag{3.17}$$

Moreover $\bar{q}'(0) = 0$.

Proof. We have seen that $\bar{q}$ is the solution of (3.16) in the sense of distributions. As $\bar{q}\in H^1(I_\alpha)$ is continuous on I_α the equation above can be extended by parity and gives the system (3.17). Since $\bar{q}\in H^1(I_\alpha)\subset L^\infty(I_\alpha)$, the function

$$\left\{\begin{array}{lcl} I_\alpha & \to & \mathbb{R}\\ r & \mapsto & \omega(r)f_{\beta,\varepsilon}(\bar{q})(r)\end{array}\right. \tag{3.18}$$

belongs to $L^2(I_\alpha)$. From (3.16) and (3.18), it can be deduced that $(\varepsilon\beta\omega\bar{q}')'\in L^2(I_\alpha)$ and $\omega\bar{q}'\in H^1(I_\alpha)$. As, $H^1(I_\alpha)\subset\mathcal{C}^0(I_\alpha)$, then $\omega\bar{q}'$ is continuous. Dividing by ω (which does not vanish), we deduce that $\bar{q}'$ is continuous on I_α. In other words, $\bar{q}\in\mathcal{C}^1(I_\alpha)$.
We use the same reasoning to prove with $q\in\mathcal{C}^1(I_\alpha)$ and relation (3.14) that $(\varepsilon\beta\omega\bar{q}')'$ is $\mathcal{C}^1$. On the other hand,

$$\left(\omega\bar{q}'\right)' = \pi H\bar{q}' + \omega\bar{q}'',$$

in the distributional sense, where H is the Heaviside function ($H\equiv -1$ on $\mathbb{R}^-$ and $H\equiv 1$ on $\mathbb{R}^+$). We noticed that $\bar{q}'(0) = 0$: therefore $\pi H\bar{q}'$ is continuous. This implies that $\omega\bar{q}''$ is continuous as well. Dividing once again by ω, we claim that $\bar{q}''$ is a continuous function. Therefore $\bar{q}\in C^2(I_\alpha)$ is a strong solution of (3.14). ∎

We now precise the solution shape. Indeed, we want it to be as close as possible to the indicator function of I_α. We are going to prove that the solution shape is as in Figure 3.3. Therefore we have to estimate $\bar{q}(0)$ and $\bar{q}'(\frac{\alpha}{2})$ to tune parameters β and ε so that $\bar{q}(0)$ is as close as possible to 1 and $|\bar{q}'(\frac{\alpha}{2})|$ as large as possible.

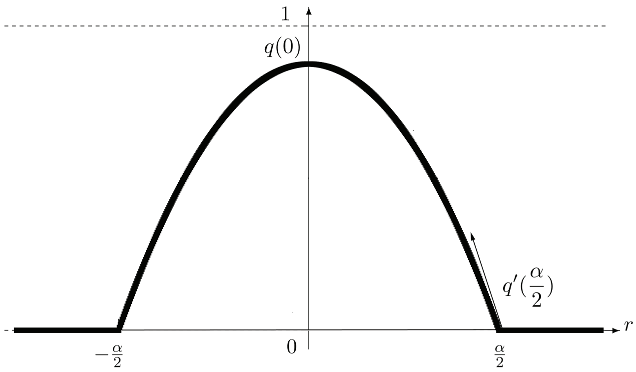


FIGURE 3.3. Solution for $N = 2$

Theorem 3.7. *Assume $N = 2$ and $\beta \le \varepsilon$. Let $\bar{p}$ be the unique solution to $(\mathcal{P}_{\varepsilon,\alpha})$and $\bar{q} = \Theta(\bar{p})$. Then*

(1) *$\bar{q}$ is an even function that is decreasing on $\left[0, \frac{\alpha}{2}\right]$,*

(2)
$$\bar{q}(0) - \frac{1}{36\beta\varepsilon}t^2 + \circ(t^2) \le \bar{q}(t) \le \bar{q}(0) \ , \tag{3.19}$$

(3)
$$- f_{\beta,\varepsilon}(\bar{q}(0))\frac{\alpha^2}{144\beta\varepsilon} \le \bar{q}(0) \le \frac{\alpha^2}{144\beta\varepsilon} \ , \tag{3.20}$$

(4)
$$- \frac{\alpha}{36\beta\varepsilon} \le \bar{q}'(\frac{\alpha}{2}) \le f_{\beta,\varepsilon}(\bar{q}(0))\frac{\alpha}{36\beta\varepsilon} \ , \tag{3.21}$$

(5)
$$q''(\frac{\alpha}{2}) \le 0 \ ,$$

where $f_{\beta,\varepsilon}$ is given by (3.8).

Proof. The proof is given in appendix. □

The previous theorem allows to tune parameters with respect to the width α, so that the solution is as close as possible of the indicator of I_α. If we want the solution to be very *flat* at 0 it is sufficient to set

$$\beta\varepsilon >> \frac{1}{36} \ .$$

with (3.19). We would like $\bar{q}(0)$ to be close to 1. Passing to the limit as $t \to 1$ in relation (3.19) gives a necessary condition

$$\alpha^2 \ge 144\beta\varepsilon \ .$$

3.6. **3D case.** Using the same techniques as in the 2D case, one can prove that the solution $\bar{q}$ is $\mathcal{C}^2$ on $]0, \frac{\alpha}{2}]$ and is a strong solution to

$$\left\{ \begin{array}{l} -9\beta\varepsilon\,(\omega\bar{q}')' + \omega f_{\beta,\varepsilon}(\bar{q}) = 0 \text{ in } [\eta, \frac{\alpha}{2}]\ , \\ \bar{q}(\eta) \text{ given }, \ \bar{q}(\frac{\alpha}{2}) = 0\ , \end{array} \right. \tag{3.22}$$

for every $\eta \in]0, \frac{\alpha}{2}]$. Nevertheless, one cannot conclude that $\bar{q}$ is $\mathcal{C}^1$ on $[0, \frac{\alpha}{2}]$ because $\omega(0) = 0$ and the solution may be singular at 0. Thus, the analogous problem to 3.16 cannot be pointless defined up to 0. Indeed, we only have a weak formulation that does allow to set appropriate boundary conditions. However, we may use the following strategy.

- The 3D problem is equivalent to a 2D one with the same projection technique. A weight function ω_2 appear with $\omega_2(0) \neq 0$.
- Then, we pass from 2D to 1D by noticing that the 2D problem owns symmetry properties once again.

We will not detail this strategy and rather present a slightly modified model that allows to give regularity results in the 3D case.

4. A modified 3D model

The possible singularity of the solution at 0 comes from the fact that $\omega(0) = 0$. This is due to the *ends* of the tube contribution to ω. Therefore, we consider a modified tube model where ends are excluded.

4.1. **Modeling the tube.** As in section 3., assumptions $(\mathcal{H}_\Gamma)$ are needed to define a Frenet-Serret frame. *Thickness* around Γ is defined now: it is said that the section of the tube along Γ is less than α (width of the tube) if the points are at a distance less than $\frac{\alpha}{2}$ of Γ. Once again it is required that the radius of curvature is not too small and we assume (3.3). We have a result similar to the one of section 3.

Proposition 4.1. *Assume $(\mathcal{H}_\Gamma)$ and (3.3) are satisfied. Then the following application Φ is a local diffeomorphism:*

$$\begin{array}{rcl} \Phi & : &]0,\ell[\times \left]0, \dfrac{\alpha}{2}\right[\times]-\pi,\pi[\to \Omega \\ (t,r,\theta) & \mapsto & F(t) + r\cos(\theta)N(t) + r\sin(\theta)B(t). \end{array}$$

The Jacobian of Φ is

$$J\Phi(t,r,\theta) \quad = \quad r(1 - r\cos(\theta)\gamma(t)).$$

The proof is similar to the one of proposition 3.2.

Definition 4.1. *Let be Φ as in proposition 4.1 and T_α the image of Φ. We say T_α is a tube if Φ is a global diffeomorphism from $]0,\ell[\times \left]0, \frac{\alpha}{2}\right[\times]-\pi,\pi[$ onto T_α.*
The (new) space $\mathcal{F}_\alpha$ is the subspace of $H^1_0(\Omega)$ with functions p such that the support of p is included in T_α and

$$a.e.\ t \in]0,\ell[,\ a.e.\ r \in \left]0, \frac{\alpha}{2}\right[,\ a.e.\ (\theta,\tilde{\theta}) \in]-\pi,\pi[^2, \quad p(\Phi(t,r,\theta)) = p(\Phi(t,r,\tilde{\theta}). \tag{4.1}$$

We now consider a minimization problem to detect such a tube T_α:

$$\min_{p\in\mathcal{F}_\alpha} \mathcal{E}_\varepsilon(p), \tag{$\mathcal{Q}_{\varepsilon,\alpha}$}$$

where

$$\mathcal{E}_\varepsilon(p) = \frac{1}{2}\int_{T_\alpha} (p-1)^2\,dx + \beta\int_{T_\alpha}\left(9\varepsilon|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon}\right)dx. \tag{4.2}$$

The following result can be proved as in section 3.1.4 .

Theorem 4.1. *Problem $(\mathcal{Q}_{\varepsilon,\alpha})$ has at least a solution. This solution is unique if $\beta \le \varepsilon$. Moreover, any solution $\bar p$ of $(\mathcal{P}_{\varepsilon,\alpha})$ satisfies*

$$\forall \varphi \in \mathcal{F}_\alpha, \quad \int_{T_\alpha} (9\beta\varepsilon \nabla \bar p \nabla \varphi + f_{\beta,\varepsilon}(p)\varphi)\, dx = 0 \ , \tag{4.3}$$

where $f_{\beta,\varepsilon}$ is defined with (3.8).

4.2. **3D tube *rectification*.** The purpose of this section is to show that with the symmetry assumptions (4.1) the minimization problem is equivalent to minimizing the functional on the set of zero curvature tubes $T^\star_\alpha$, i.e. the ones for which Γ is a segment.

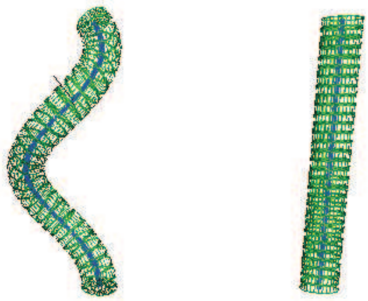


FIGURE 4.1. Tube *rectification*

Definition 4.2. *To consider the rectified problem we set :*

- $\Gamma^\star$ *the $\mathbb{R}^3$-segment given by*

$$\begin{array}{rcl} F^\star : [0,\ell] & \to & \mathbb{R}^3 \\ t & \to & (t,0,0), \end{array}$$

- $T^\star_\alpha$ *the tube associated to $\Gamma^\star$ of width α,*

$$T^\star_\alpha = \{x \in \mathbb{R}^3 \mid x = (t, r\cos\theta, r\sin\theta),\ t \in [0,\ell],\ r \in [0, \frac{\alpha}{2}],\ \theta \in [-\pi,\pi]\ \},$$

- $\mathcal{F}^\star_\alpha \subset H^1_0(T^\star_\alpha)$ *the space of functions q with support in $T^\star_\alpha$ such that for almost every $(t,r) \in [0,\ell] \times \left[0,\frac{\alpha}{2}\right]$ and $(\theta_1,\theta_2) \in \mathbb{R}^2$:*

$$q(t, r\cos\theta_1, r\sin\theta_1) = q(t, r\cos\theta_2, r\sin\theta_2),$$

- $\omega : T^\star_\alpha \to \mathbb{R}$ *such that*

$$\omega(t, r\cos\theta, r\sin\theta) = \frac{1}{1 - r^2(\cos\theta)^2\gamma(t)^2},$$

- $\mathbb{A}_\omega : T^\star_\alpha \to M_{3,3}(\mathbb{R})$ *such that*

$$\mathbb{A}_\omega(t, r\cos\theta, r\sin\theta) = \begin{pmatrix} \sqrt{\omega(t, r\cos\theta, r\sin\theta)} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.4}$$

- $\mathcal{E}^\star_\varepsilon$ *the energy functional defined on $\mathcal{F}^\star_\alpha$ by*

$$\mathcal{E}^\star_\varepsilon(q) = \frac{1}{2}\int_{T^\star_\alpha} (q-1)^2\, dx + \beta \int_{T^\star_\alpha} \left(9\varepsilon|A\nabla q|^2 + \frac{(q(q-1))^2}{\varepsilon}\right) dx, \tag{4.5}$$

- $H^1_\omega(T^\star_\alpha)$ *the space of measurable functions such that*

$$\|p\|^2_{H^1_\omega} = \int_{T^\star_\alpha} (|\mathbb{A}_\omega \nabla p|^2 + p^2)\, dx < +\infty.$$

Let $(\mathcal{Q}^\star_{\varepsilon,\alpha})$ be the minimization problem

$$\min_{q\in\mathcal{F}^\star_\alpha} E^\star_\varepsilon(q). \tag{$\mathcal{Q}^\star_{\varepsilon,\alpha}$}$$

With assumptions of proposition 4.1 and (3.3) we get :

$$0 \le r \le \frac{\alpha}{2}, t\in[0,\ell] \quad \Rightarrow \quad 0\le r\gamma(t) < \frac{1}{2}.$$

Therefore the application ω makes sense and takes its values in $[1,\frac{4}{3}]$. This proves that

$$H^1(T^\star_\alpha) = H^1_\omega(T^\star_\alpha)\ .$$

Now we may define Θ as

$$\begin{array}{rcl}\Theta : T_\alpha & \to & T^\star_\alpha \\ x & \mapsto & (t, r\cos\theta, r\sin\theta),\end{array}$$

where $x=\Phi(t,r,\theta)$. As Φ is a diffeomorphism (proposition 4.1) and the polar coordinates parametrization as well, then Θ is also a diffeomorphism.

Proposition 4.2. *The following application is an isomorphism :*

$$\begin{array}{rcl}\Psi : \mathcal{F}_\alpha & \to & \mathcal{F}^\star_\alpha \\ p & \mapsto & p\circ\Theta.\end{array}$$

Moreover, $\|p\|_{H^1(\Omega)} = \|\Psi(p)\|_{H^1_\omega(T^\star_\alpha)}$ *and* $\mathcal{E}_\varepsilon(p) = \mathcal{E}^\star_\varepsilon(\Psi(p))$.

Proof. The proof is given in appendix. □

We have the final result:

Theorem 4.2. *Problems* $(\mathcal{Q}_{\varepsilon,\alpha})$ *and* $(\mathcal{Q}^\star_{\varepsilon,\alpha})$ *are equivalent. More precisely* p *is the unique solution to* $(\mathcal{Q}_{\varepsilon,\alpha})$ *if and only if* $\Psi(p)$ *is the unique solution to* $(\mathcal{Q}^\star_{\varepsilon,\alpha})$.

4.3. **Solution regularity.** We end this section by giving regularity properties of $(\mathcal{Q}_{\varepsilon,\alpha})$ solution. For this, we first show that the solution of the problem $(\mathcal{Q}^\star_{\varepsilon,\alpha})$ is solution of a more general problem. Then we will use Theorem 4.2 to conclude. Consider

$$\min_{q\in H^1_0(T^\star_\alpha)} \mathcal{E}^\star_\varepsilon(q). \tag{$\mathcal{Q}^{\star\star}_{\varepsilon,\alpha}$}$$

It is now classical to see that if $\beta\le\varepsilon$ then problem $(\mathcal{Q}^{\star\star}_{\varepsilon,\alpha})$ has a unique solution.

Theorem 4.3. *Assume* $\beta\le\varepsilon$ *and let* $p^\star$ *be the solution of* $(\mathcal{Q}^{\star\star}_{\varepsilon,\alpha})$. *Then* $p^\star\in\mathcal{C}^\infty(T_\alpha)\cap\mathcal{F}^\star_\alpha$.

Proof. We first prove that any solution p of $(\mathcal{Q}^{\star\star}_{\varepsilon,\alpha})$ belongs to $\mathcal{C}^\infty(T^\star_\alpha)$. The first order optimality condition gives

$$\forall\varphi\in H^1_0(T^\star_\alpha),\qquad \int_{T^\star_\alpha}(9\beta\varepsilon\nabla p\nabla\varphi + f_{\beta,\varepsilon}(p)\varphi)\,dx = 0\ , \tag{4.6}$$

where $f_{\beta,\varepsilon}$ is given by (3.8). Then

$$9\beta\varepsilon\Delta p = f_{\beta,\varepsilon}(p) \text{ on } T^\star_\alpha,$$

in the sense of distributions. As $T^\star_\alpha$ is a smooth open subset of $\mathbb{R}^3$, then $H^1_0(T_\alpha)\subset L^6(T_\alpha)$([23] for example). This implies that $f_{\beta,\varepsilon}(p)\in L^2(T^\star_\alpha)$. Thanks to the ellipticity of the Laplacian operator we deduce that p is C^∞.
Let us show now that $p^\star\in\mathcal{F}^\star_\alpha$. We use a symmetrization technique. Let H be an

hyperplane including $\Gamma^\star$, Π the orthogonal symmetry with respect to H and set $\tilde{p} := p \circ \Pi$. As problem $(\mathcal{Q}^{\star\star}_{\varepsilon,\alpha})$ is invariant by the rotations of axis $\Gamma^\star$, the function $\tilde{p}$ is solution of the problem as well. By uniqueness, $\tilde{p} = p^\star$. this proves that $p^\star$ has a cylindrical symmetry of axis $\Gamma^\star$. □

We may now conclude:

Theorem 4.4. *If $\beta \leq \varepsilon$, the unique solution to $\bar{p}$ de $(\mathcal{Q}_{\varepsilon,\alpha})$ belongs to $\mathcal{C}^2(T_\alpha)$.*

Proof. Theorem 4.3 tells that the unique solution $p^\star$ of $(\mathcal{Q}^\star_{\varepsilon,\alpha})$ belongs to $\mathcal{C}^\infty(T^\star_\alpha)$. In addition, $\bar{p} = \Psi^{-1}(p^\star)$ is the unique solution of $(\mathcal{Q}_{\varepsilon,\alpha})$. As $\Psi^{-1}(p^\star)$ as the same regularity as Φ, we deduce that $\bar{p}$ belongs to $\mathcal{C}^2(T_\alpha)$ since Φ is $\mathcal{C}^2$. □

5. Conclusion

The model we have presented allows to consider thin structures segmentation via a geometrical prior. However, the model is too general and we have to make it more precise. Next step is to consider the case where the tube width α is not constant any longer. Using the same techniques, we infer that we will get the same kind of results. This will be addressed in a future work.

In addition, the angiography network we have to recover is not made of isolated tubes. We have to deal with junctions : this is a more technical work since the local parametrization with the Frenet-Serret frame is not straightforward. A different point of view is to consider the network as the solution of a shape optimization problem involving the behavior of the blood as a Navier-Stokes fluid.

Last but not least, we actually perform numerical simultations and different tests with respect to the parameters α, β and ε. This will be reported in a near future.

6. Appendix : Proofs

6.1. Proof of Lemma 3.2.

2D case . As $\ell \leq \omega(r) \leq \ell + \pi\frac{\alpha}{2}$, for every $r \in I_\alpha$, we get

$$\ell \|q\|^2_{H^1(I_\alpha)} \leq \|q\|^2_{H^1_\omega(I_\alpha)} \leq (\ell + \pi\frac{\alpha}{2})\|q\|^2_{H^1(I_\alpha)}.$$

3D case. Let be $q \in H^1_\omega(I_\alpha)$ and $r \in \left]0, \frac{\alpha}{2}\right]$. Let us prove that q is continuous on $I_\alpha \setminus I_r$. Choose $r' \in]0, r[$ and ν a $C^1(I_\alpha)$ function identically equal to 1 on $I_\alpha \setminus I_r$ with support in $I_\alpha \setminus I_{r'}$. The function νq belongs to $H^1(I_\alpha)$. Indeed,

$$\begin{aligned}
\int_{I_\alpha} ((\nu q)')^2\,dr = \int_{I_\alpha} (\nu' q + \nu q')^2\,dr &\leq 2\int_{I_\alpha} \left((\nu' q)^2 + (\nu q')^2\right)\,dr\ ,\\
&\leq 2\|\nu'\|_\infty \left(\int_{I_\alpha\setminus I_{r'}} q^2\,dr + \int_{I_\alpha\setminus I_{r'}} (q')^2\,dr\right)\ ,\\
&\leq \frac{2\|\nu'\|_\infty}{\omega(r')} \left(\int_{I_\alpha\setminus I_{r'}} \omega q^2\,dr + \int_{I_\alpha\setminus I_{r'}} \omega(q')^2\,dr\right)\ ,\\
&< +\infty
\end{aligned}$$

and

$$\begin{aligned}
\int_{I_\alpha} (\nu q)^2\,dr &\leq \|\nu\|^2_\infty \int_{I_\alpha\setminus I_{r'}} q^2\,dr \leq \frac{\|\nu\|^2_\infty}{\omega(r')} \int_{I_\alpha\setminus I_{r'}} \omega(r) q^2(r)\,dr\ ,\\
&\leq \frac{\|\nu\|^2_\infty}{\omega(r')} \int_{I_\alpha} \omega(r) q^2(r)\,dr < +\infty.
\end{aligned}$$

Thus νq is a continuous function on I_α. As $\nu \equiv 1$ on $I_\alpha \setminus [-r,r]$, then q is continuous on $I_\alpha \setminus [-r,r]$ for every $r > 0$. Therefore q is continuous on $I_\alpha \setminus \{0\}$. □

6.2. **Proof of Proposition 3.3.**

(i) Let us show relation (3.11) for $N = 2$. For every $p \in \mathcal{F}_\alpha$, we have

$$\|p\|^2_{H^1(\Omega)} = \iint_\Omega \left(|p|^2 + |\nabla p|^2\right) dx = \iint_{A_\alpha} \left(|p|^2 + |\nabla p|^2\right) dx.$$

$$\begin{aligned}
\|p\|^2_{H^1(\Omega)} \quad = \quad & \underbrace{\int_{t=0}^{\ell} \int_{r=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C(t,r) |1 - r\gamma(t)|\, dr\, dt}_{\textit{Body } \mathcal{C}_\alpha} \\
& + \underbrace{\int_{\theta=0}^{\pi} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_{B^0}(r,\theta) r\, dr\, d\theta}_{\textit{End } B^0} \\
& + \underbrace{\int_{\theta=0}^{\pi} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_{B^\ell}(r,\theta) r\, dr\, d\theta}_{\textit{End } B^\ell},
\end{aligned}$$

• Let us estimate

$$I_1 := \int_{t=0}^{\ell} \int_{r=-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C(t,r) |1 - r\gamma(t)|\, dr\, dt.$$

Assumption (3.3) yields that

$$\forall r \in I_\alpha,\ \forall t \in [0,\ell], \quad 1 - r\gamma(t) > 0.$$

As $p \in \mathcal{F}_\alpha$, it is an even function with respect to r. Thus , we get

$$\begin{aligned}
I_1 \quad = \quad & \int_{t=0}^{\ell} \int_{r=-\frac{\alpha}{2}}^{0} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C(t,r)(1 - r\gamma(t))\, dr\, dt \\
& + \int_{t=0}^{\ell} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C(t,r)(1 + r\gamma(t))\, dr\, dt\ , \\
I_1 \quad = \quad & 2 \int_{t=0}^{\ell} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C(t,r)\, dr\, dt.
\end{aligned}$$

Thanks to definition 3.2, we get

$$\forall (t,r) \in [0,\ell] \times I_\alpha, \quad p(\Phi_C(t,r)) = p(\Phi_C(0,r)).$$

Differentiating with respect to t, gives

$$\forall r \in I_\alpha, \quad (1 - r\gamma(t))\ \langle \nabla p(\Phi_C(t,r)), T(t)\rangle = 0.$$

As $1 - r\gamma(t) > 0$ then $\nabla p(\Phi_C(t,\cdot))$ is orthogonal to $T(t)$ and, thus, colinear to $N(t)$. This gives

$$|\nabla p|^2 \circ \Phi_C(t,r) = |\langle \nabla p(\Phi_C(t,r)), N(t)\rangle|^2.$$

Since $q(r) = p(F(t) + rN(t))$ then $q'(r) = \langle \nabla p(\Phi_C(t,r)), N(t)\rangle$.
We finally obtain $q'(r)^2 = |\nabla p|^2 \circ \Phi_C(t,r)$ and

$$I_1 = 2\ell \int_{r=0}^{\frac{\alpha}{2}} \left(|q(r)|^2 + |q'(r)|^2\right) dr. \tag{6.1}$$

• We notice that $\Phi_{B^0}(r,\theta) = \Phi_{B^0}(r,0)$, so that

$$I_2 = \int_{\theta=0}^{\pi} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \Phi_{B^0}(r,\theta) r \, dr \, d\theta,$$

writes

$$I_2 = \int_{\theta=0}^{\pi} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \Phi_{B^0}(r,0) r \, dr \, d\theta = \pi \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \Phi_{B^0}(r,0) r \, dr \, d\theta.$$

As $\Phi_{B^0}(r,\theta) = F(0) + r\cos(\theta)N(0) - r\sin(\theta)T(0) = r$, the function $\theta \to p(F(0) + r\cos(\theta)N(0) - r\sin(\theta))$ is constant on $[0,\pi]$. The differentiation gives

$$\forall \theta \in [0,\pi], \qquad \langle \nabla p \circ \Phi_{B^0}(r,\theta), (-\sin(\theta)N(0) - \cos(\theta)T(0)) \rangle = 0.$$

This means that $\nabla p \circ \Phi_{B^0}(r,\theta)$ is orthogonal to $-\sin(\theta)N(0) - \cos(\theta)T(0)$ and colinear to $\cos(\theta)N(0) - \sin(\theta)T(0)$. In addition, $q(r) = p \circ \Phi_{B^0}(r,\theta)$, which yields

$$|q'(r)| = |\langle \nabla\nabla p \circ \Phi_{B^0}(r,\theta), \cos(\theta)N(0) - \sin(\theta)T(0) \rangle|.$$

As $\cos(\theta)N(0) - \sin(\theta)T(0)$ is a unit vector we get

$$|q'(r)| = |\nabla p(F(0) + r\cos(\theta)N(0) - r\sin(\theta)T(0)|.$$

Eventually, we have

$$I_2 = \pi \int_{r=0}^{\frac{\alpha}{2}} \left(|q|^2 + |q^{'}|^2\right) r dr. \tag{6.2}$$

• We can prove similarly that

$$I_3 := \int_{\theta=0}^{\pi} \int_{r=0}^{\frac{\alpha}{2}} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_{B^\ell}(r,\theta) r \, dr \, d\theta$$

verifies

$$I_3 = \pi \int_{r=0}^{\frac{\alpha}{2}} \left(|q|^2 + |q^{'}|^2\right) r dr. \tag{6.3}$$

• The above estimates give

$$\|p\|^2_{H^1(\Omega)} \quad = \quad \int_{r=0}^{\frac{\alpha}{2}} (2\pi r + 2\ell) \left(|q|^2 + |q^{'}|^2\right) dr.$$

As q is an even function

$$\|p\|^2_{H^1(\Omega)} \quad = \quad \int_{I_\alpha} \left(|q|^2 + |q^{'}|^2\right) \omega(r) \, dr,$$

that is $\|p\|_{H^1(\Omega)} = \|q\|_{H^1_\omega(I_\alpha)}$.
(ii) We can show equality (3.11) similarly fo $N = 3$: however

1. we deal with triple integrals,
2. the jacobian of Φ_C at (t,r,θ) is $|r(1 - r\cos(\theta)\gamma(t))|$,
3. the jacobian of Φ_{B^0}, Φ_{B^ℓ} at (r,θ,ϕ) is $r^2\cos(\phi)$.

We set $D_C = [0,\ell] \times I_\alpha \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and $D_B = [0, \frac{\alpha}{2}] \times [0, 2\pi] \times \left[0, \frac{\pi}{2}\right]$; then

$$\begin{aligned}
\|p\|^2_{H^1(\Omega)} &= \underbrace{\iiint_{(t,r,\theta)\in D_c} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C |r(1 - r\cos(\theta)\gamma(t))|}_{I_1 := \ Body} \\
&\quad + \underbrace{\iiint_{(r,\theta,\phi)\in D_B} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_{B^0} r^2 \cos(\phi)}_{I_2 := \ end\ B^0} , \\
&\quad + \underbrace{\iiint_{(r,\theta,\phi)\in D_B} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_{B^\ell} r^2 \cos(\phi)}_{I_3 := \ end\ B^\ell} ,
\end{aligned}$$

As previously we get with (3.3)

$$\forall r \in I_\alpha, \forall t \in 0, \ell], \quad |r(1 - r\cos(\theta)\gamma(t))| = |r|(1 - r\cos(\theta)\gamma(t)).$$

As p is an even function with respect to r we obtain

$$\begin{aligned}
I_1 &= \iiint_{(t,r,\theta)\in D_C^+} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C |r|(1 - r\cos(\theta)\gamma(t))\, dt\, dr\, d\theta \\
&\quad + \iiint_{(t,r,\theta)\in D_C^+} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C |r|(1 + r\cos(\theta)\gamma(t))\, dt\, dr\, d\theta , \\
I_1 &= 2 \iiint_{(t,r,\theta)\in D_C^+} \left(|p|^2 + |\nabla p|^2\right) \circ \Phi_C |r|.
\end{aligned}$$

where $D_C^+ = D_C \cap \{r \geq 0\}$. With symmetry arguments we deduce

$$I_1 = \ell\pi \int_{I_\alpha} \left(|q|^2 + |q'|^2\right) |r|\, dr.$$

We prove as in the 2D-case that

$$I_2 = I_3 = \pi \int_{I_\alpha} \left(|q|^2 + |q'|^2\right) r^2\, dr.$$

and

$$\|p\|^2_{H^1(\Omega)} = \int_{I_\alpha} (\ell\pi|r| + 2\pi r^2) \left(|q|^2 + |q'|^2\right) dr.$$

Equality (3.11) holds for $N = 2, 3$. This implies that Θ is an application from $\mathcal{F}_\alpha$ to $H^1_{\omega,0}(I_\alpha)$. Moreover, with definition 3.2, $\Theta(p)$ is an even function that vanishes on I_α boundary, for every $p \in \mathcal{F}_\alpha$. More precisely, $\Theta(\mathcal{F}_\alpha) \subset \mathcal{G}^\omega_\alpha$. The bijectivity of Θ comes from definition 3.2. □

6.3. **Proof of Proposition 3.4.**

Let be $q \in \mathcal{G}^\omega_\alpha$ and $p = \Theta^{-1}(q) \in \mathcal{F}_\alpha$, we have

$$\begin{aligned}
G_\varepsilon(q) = \mathcal{E}_\varepsilon(p) &= \frac{1}{2}\int_\Omega (p - \chi_{A_\alpha})^2\, dr + \beta \int_\Omega 9\varepsilon|\nabla p|^2 + \frac{(p(p-1))^2}{\varepsilon}\, dr, \\
&= \frac{1}{2}\int_{A_\alpha} (p - 1)^2\, dr + \beta \int_\Omega 9\varepsilon|\nabla p|^2 + \frac{(p(p-1))^2}{\varepsilon}\, dr.
\end{aligned}$$

The computation is similar to the previous ones. We obtain (2D case)

$$\begin{aligned} G_\varepsilon(q) &= \ell \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} 9\beta\varepsilon|q'|^2 + \frac{1}{2}(1-q)^2 + \beta\frac{(q-q^2)^2}{\varepsilon}\,dr \\ &\quad + \pi \int_{r=0}^{\frac{\alpha}{2}} \left(18\beta\varepsilon|q'|^2 + (1-q)^2 + 2\beta\frac{(q-q^2)^2}{\varepsilon}\right) r\,dr\ , \\ &= \int_{-\frac{\alpha}{2}}^{\frac{\alpha}{2}} \left(9\beta\varepsilon|q'|^2 + \frac{1}{2}(1-q)^2 + \beta\frac{(q-q^2)^2}{\varepsilon}\right)(\ell+\pi|r|)\,dr\ . \end{aligned}$$

The 3D computation is quite similar. □

6.4. **Proof of Theorem 3.7.**

Let $\bar p$ be the unique solution to $(\mathcal{P}_{\varepsilon,\alpha})$and $\bar q = \Theta(\bar p)$. We denote $q = \bar q_{|[0,\frac{\alpha}{2}]}$ the solution to problem (3.13).

(1) As $\bar q$ is even, it is sufficient to prove that q is decreasing on $[0,\frac{\alpha}{2}]$. We have proved that q is the strong solution of

$$9\beta\varepsilon\left(\omega q'\right)' = \omega f_{\beta,\varepsilon}(q) \text{ in } [0,\frac{\alpha}{2}].$$

As $q \le 1$ and $f_{\beta,\varepsilon}$ is increasing with $f_{\beta,\varepsilon}(1) = 0$ we get

$$\forall r \in [0,\frac{\alpha}{2}], \quad \omega(r) f_{\beta,\varepsilon}(q)(r) \le 0.$$

Therefore $(\omega q')'$ is a continuous, negative function on $[0,\frac{\alpha}{2}]$ and $\omega q'$ is decreasing. In particular

$$\forall r \in [0,\frac{\alpha}{2}], \quad \omega(r) q'(r) \le \omega(0) q'(0) = 0.$$

Thus $q' \le 0$ and q is decreasing.

(2) Let us perform a local study at $t=0$. With equation (3.16) we have

$$\forall t \in [0,\frac{\alpha}{2}], \quad 9\beta\varepsilon \int_0^t \left(\omega(s)\bar q'(s)\right)' ds = \int_0^t \omega(s) f_{\beta,\varepsilon}(\bar q(s))\,ds\ .$$

Let us estimate

$$\int_0^t \omega(s) f_{\beta,\varepsilon}(\bar q(s))\,ds\ .$$

As $\bar q$ is decreasing on $[0,t]$, takes its values in $[0,1]$ and $f_{\beta,\varepsilon}$ is an increasing, negative function, we have: $\forall 0 \le s \le t \le \frac{\alpha}{2}$

$$-\frac{1}{2} = f_{\beta,\varepsilon}(0) \le f_{\beta,\varepsilon}(\bar q(t)) \le f_{\beta,\varepsilon}(\bar q(s)) \le f_{\beta,\varepsilon}(\bar q(0)) \le 0 \tag{6.4}$$

and (with $\omega \ge 0$)

$$f_{\beta,\varepsilon}(\bar q(t)) \int_0^t \omega(s)\,ds \le \int_0^t \omega(s) f_{\beta,\varepsilon}(\bar q(s))\,ds \le f_{\beta,\varepsilon}(\bar q(0)) \int_0^t \omega(s)\,ds\ \le 0,$$

that is

$$f_{\beta,\varepsilon}(\bar q(t)) \frac{(\ell+\pi t)^2 - \ell^2}{2\pi} \le 9\beta\varepsilon\omega(t)\bar q'(t) \le f_{\beta,\varepsilon}(\bar q(0)) \frac{(\ell+\pi t)^2 - \ell^2}{2\pi},$$

since $\bar q'(0) = 0$. Thus we obtain

$$f_{\beta,\varepsilon}(\bar q(t)) \frac{(2\ell+\pi t)t}{2} \le 9\beta\varepsilon(\ell+\pi t)\bar q'(t) \le f_{\beta,\varepsilon}(\bar q(0)) \frac{(2\ell+\pi t)t}{2},$$

that is finally

$$\forall t \in [0, \frac{\alpha}{2}], \quad \lambda(t)\, f_{\beta,\varepsilon}(\bar{q}(t))\frac{t}{18\beta\varepsilon} \le \bar{q}'(t) \le \lambda(t)\, f_{\beta,\varepsilon}(\bar{q}(0))\frac{t}{18\beta\varepsilon} \ , \tag{6.5}$$

where we have set

$$\lambda(t) = \frac{(2\ell + \pi t)}{\ell + \pi t} = 1 + \frac{\ell}{\ell + \pi t} \ . \tag{6.6}$$

By continuity, we get

$$\bar{q}\text{"}(0) = \lim_{t \to 0^+} \frac{\bar{q}(t)}{t} = \frac{f_{\beta,\varepsilon}(\bar{q}(0))}{9\beta\varepsilon} \ ,$$

and a local expansion of $\bar{q}$ at 0 as well:

$$\bar{q}(t) = \bar{q}(0) + \frac{f_{\beta,\varepsilon}(\bar{q}(0))}{18\beta\varepsilon} t^2 + \circ(t^2) \ ,$$

since $q'(0) = 0$. As $0 \ge f_{\beta,\varepsilon}(\bar{q}(0)) \ge -\frac{1}{2}$ we obtain inequality (3.19).

(3) Equations (6.4), (6.5) and $1 \le \lambda(t) \le 2$ give

$$\forall t \in [0, \frac{\alpha}{2}], \quad -1 \le \lambda(t)\, f_{\beta,\varepsilon}(\bar{q}(t)) \text{ and } \lambda(t)\, f_{\beta,\varepsilon}(\bar{q}(0)) \le \ f_{\beta,\varepsilon}(\bar{q}(0)) \ .$$

This yields

$$\forall t \in [0, \frac{\alpha}{2}], \quad -\frac{t}{18\beta\varepsilon} \le \bar{q}'(t) \le f_{\beta,\varepsilon}(\bar{q}(0))\frac{t}{18\beta\varepsilon} \ .$$

Performing an integration between 0 and $\frac{\alpha}{2}$ gives

$$-\frac{\alpha^2}{8 * 18\beta\varepsilon} \le \bar{q}(\frac{\alpha}{2}) - \bar{q}(0) \le f_{\beta,\varepsilon}(\bar{q}(0))\frac{\alpha^2}{8 * 18\beta\varepsilon} \ ,$$

and with $\bar{q}(\frac{\alpha}{2}) = 0$:

$$-f_{\beta,\varepsilon}(\bar{q}(0))\frac{\alpha^2}{144\beta\varepsilon} \le \bar{q}(0) \le \frac{\alpha^2}{144\beta\varepsilon} \ .$$

(4) $q'(\frac{\alpha}{2})$ estimate. Equation (6.5) with $t = \dfrac{\alpha}{2}$ provides

$$2\, f_{\beta,\varepsilon}(\bar{q}(\frac{\alpha}{2}))\frac{\alpha}{2 * 18\beta\varepsilon} \le \bar{q}'(\frac{\alpha}{2}) \le f_{\beta,\varepsilon}(\bar{q}(0))\frac{\alpha}{2 * 18\beta\varepsilon}$$

since $1 \le \lambda(t) \le 2$. As $f_{\beta,\varepsilon}(\bar{q}(\dfrac{\alpha}{2})) = f_{\beta,\varepsilon}(0) = -\dfrac{1}{2}$ we have proved relation (3.21).

(5) We finally prove that $q''(\frac{\alpha}{2}) \le 0$. The differential equation writes

$$\forall t \in]0, \frac{\alpha}{2}[, \quad -9\beta\varepsilon\pi\bar{q}'(t) - 9\beta\varepsilon(\ell + \pi t)\bar{q}\text{"}(t) + (\ell + \pi t) f_{\beta,\varepsilon}(\bar{q}(t)) = 0 \ .$$

Passing to the limit as $t \to \frac{\alpha}{2}$, we obtain

$$\bar{q}\text{"}(\frac{\alpha}{2}) = -\frac{\pi}{\ell + \pi\frac{\alpha}{2}}\bar{q}'(\frac{\alpha}{2}) - \frac{1}{18\beta\varepsilon}(\ell + \pi\frac{\alpha}{2}) \ .$$

Equation (3.21) yields

$$0 \le -\bar{q}'(\frac{\alpha}{2}) \le \frac{\alpha}{36\beta\varepsilon} \ .$$

So, we may conclude

$$\bar{q}\text{"}(\frac{\alpha}{2}) \le \frac{\alpha\pi}{36\beta\varepsilon}\frac{1}{(\ell + \pi\frac{\alpha}{2})} - \frac{1}{18\beta\varepsilon} \le \frac{-2\ell}{36\beta\varepsilon(\ell + \pi\frac{\alpha}{2})} \le 0 \ .$$

□

6.5. **Proof of Proposition 4.2.**
Let be $p \in \mathcal{F}_\alpha$. As the support of p is included in T_α, we have

$$\|p\|^2_{H^1(\Omega)}\, dx = \iiint_\Omega (|p|^2 + |\nabla p|^2)\, dx = \iiint_{T_\alpha} (|p|^2 + |\nabla p|^2)\, dx.$$

We know that Φ is a parametrization of T_α. Denoting $D =]0,\ell[\times \left]0, \frac{\alpha}{2}\right[\times]-\pi,\pi[$, and $\mathbf{x} = (t,r,\theta)$

$$\|p\|^2_{H^1(\Omega)} = \iiint_{\mathbf{x}\in D} \left(|p\circ\Phi|^2 + |\nabla p\circ\Phi|^2\right)\, |r(1 - r\cos(\theta)\gamma(t))|\, d\mathbf{x}.$$

With (3.3),

$$\forall r \in [0, \frac{\alpha}{2}],\ \forall t \in [0,\ell] \qquad r\gamma(t) < \frac{1}{2}$$

and

$$\forall (t,r,\theta) \in D, \quad |r(1 - r\cos(\theta)\gamma(t))| = r(1 - r\cos(\theta)\gamma(t)).$$

This yields

$$\begin{aligned} \|p\|^2_{H^1(\Omega)} &= \underbrace{\iiint_{\mathbf{x}\in D} (|p|^2\circ\Phi)\, r(1 - r\cos(\theta)\gamma(t))\, d\mathbf{x}}_{I_1} \\ &\quad + \underbrace{\iiint_{\mathbf{x}\in D} (|\nabla p|^2\circ\Phi)\, r(1 - r\cos(\theta)\gamma(t))\, d\mathbf{x}}_{I_2}. \end{aligned}$$

Let us split D as $D = D^+ \cup D^-$ with

$$D^+ = D \cap]0,\ell[\times \left]0, \frac{\alpha}{2}\right[\times \left(\left]-\frac{\pi}{2}, \frac{\pi}{2}\right[\right),\ D^- = D \setminus D^+,$$

so that $\cos\theta \geq 0$ on D^+ and $\cos\theta \leq 0$ on D^-. As $(p\circ\Phi)(t,r,\theta) = (p\circ\Phi)(t,r,-\theta)$, it comes

$$\begin{aligned} I_1 &= \iiint_{\mathbf{x}\in D^+} (|p|^2\circ\Phi)\, r(1 - r\cos(\theta)\gamma(t))\, d\mathbf{x} + \iiint_{\mathbf{x}\in D^-} (|p|^2\circ\Phi)\, r(1 - r\cos(\theta)\gamma(t))\, d\mathbf{x}, \\ &= \iiint_{\mathbf{x}\in D^+} (|p|^2\circ\Phi)\, r(1 - r\cos(\theta)\gamma(t))\, d\mathbf{x} + \iiint_{\mathbf{x}\in D^+} (|p|^2\circ\Phi)\, r(1 + r\cos(\theta)\gamma(t))\, d\mathbf{x}, \\ &= 2\iiint_{(t,r,\theta)\in D^+} (|p|^2\circ\Phi)\, r\, dr\, dt\, d\theta = \iiint_{(t,r,\theta)\in D} (|p|^2\circ\Phi)\, r\, dr\, dt\, d\theta. \end{aligned}$$

The computation of I_2 is different because $|(\nabla p\circ\Phi)(t,r,\theta)| = |(\nabla p\circ\Phi)(t,r,-\theta)|$ may not be true. Indeed if the tube curvature is zero then a cylindrical symmetric function has a gradient whose norm is also symmetric. This is not true any longer if the tube curvature is not zero. Set $q = p\circ\Phi$ so that

$$q(t,r,\theta) = p(F(t) + r\cos(\theta)N(t) + r\sin(\theta)B(t)).$$

With the differential properties of the Frenet-Serret frame we get

$$\left\{\begin{aligned} \frac{\partial q}{\partial t} &= \langle \nabla p\circ\Phi, (1 - r\cos(\theta)\gamma)T - r\sin(\theta)\tau N + r\cos(\theta)\tau B\rangle\ , \\ \frac{\partial q}{\partial r} &= \langle \nabla p\circ\Phi, \cos(\theta)N + \sin(\theta)B\rangle\ , \\ \frac{\partial q}{\partial \theta} &= \langle \nabla p\circ\Phi, -r\sin(\theta)N + r\cos(\theta)B\rangle\, . \end{aligned}\right.$$

As q is symmetric $\dfrac{\partial q}{\partial \theta} = 0$. So

$$\left\{\begin{array}{rcl} \dfrac{\partial q}{\partial t} &=& \langle \nabla p \circ \Phi, (1 - r\cos(\theta)\gamma) T\rangle \ , \\ \dfrac{\partial q}{\partial r} &=& \langle \nabla p \circ \Phi, \cos(\theta) N + \sin(\theta) B\rangle \ , \\ 0 &=& (\langle \nabla p \circ \Phi, -r\sin(\theta) N + r\cos(\theta) B\rangle \, . \end{array}\right.$$

The vector $\nabla p \circ \Phi$ is always orthogonal to $-r\sin(\theta)N + r\cos(\theta)B$: therefore it belongs to the plane spanned by T and $u = \cos(\theta)N + \sin(\theta)B$. These are orthogonal vectors and Pythagorean theorem gives

$$|\nabla p|^2 \circ \Phi = |\nabla p \circ \Phi|^2 = \frac{1}{(1 - r\cos(\theta)\gamma)^2} \left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial q}{\partial r}\right)^2 .$$

With the above result I_2 writes

$$\begin{aligned} I_2 &= \iiint_{\mathbf{x} \in D} \left(\frac{1}{(1 - r\cos(\theta)\gamma)^2} \left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial q}{\partial r}\right)^2 \right) r(1 - r\cos(\theta)\gamma)\, d\mathbf{x} \\ &= \underbrace{\iiint_{\mathbf{x} \in D} \frac{r}{1 - r\cos(\theta)\gamma} \left(\frac{\partial q}{\partial t}\right)^2 d\mathbf{x}}_{I_3} + \underbrace{\iiint_{\mathbf{x} \in D} r(1 - r\cos(\theta)\gamma) \left(\frac{\partial q}{\partial r}\right)^2 d\mathbf{x}}_{I_4} . \end{aligned}$$

With $D = D^+ \cup D^-$, we get

$$\begin{aligned} I_3 &= \iiint_{(t,r,\theta) \in D^+} r \left(\frac{1}{1 - r\cos(\theta)\gamma} + \frac{1}{1 + r\cos(\theta)\gamma} \right) \left(\frac{\partial q}{\partial t}\right)^2 dr\, dt\, d\theta \\ &= \iiint_{(t,r,\theta) \in D} \frac{r}{1 - r^2(\cos(\theta))^2\gamma^2} \left(\frac{\partial q}{\partial t}\right)^2 dr\, dt\, d\theta. \end{aligned}$$

Similarly

$$\begin{aligned} I_4 &= \iiint_{(t,r,\theta) \in D+} r\,(1 - r\cos(\theta)\gamma + 1 + r\cos(\theta)\gamma) \left(\frac{\partial q}{\partial t}\right)^2 dr\, dt\, d\theta \\ &= \iiint_{(t,r,\theta) \in D} r \left(\frac{\partial q}{\partial r}\right)^2 dr\, dt\, d\theta. \end{aligned}$$

Finally

$$I_2 = \iiint_{(t,r,\theta) \in D} \left[\omega \left(\frac{\partial q}{\partial t}\right)^2 + \left(\frac{\partial q}{\partial r}\right)^2 + \left(\frac{\partial q}{\partial \theta}\right)^2 \right] r\, dr\, dt\, d\theta.$$

Using (4.4) we obtain (with $q = p \circ \Phi$)

$$I_2 = \iiint_{(t,r,\theta) \in D} |\mathbb{A}_\omega \nabla q|^2 r\, dr\, dt\, d\theta = \iiint_{(t,r,\theta) \in D} (|\mathbb{A} \nabla p|^2 \circ \Phi)\, r\, dr\, dt\, d\theta.$$

Eventually,

$$\|p\|^2_{H^1(\Omega)} = \iiint_{(t,r,\theta) \in D} [(|\mathbb{A}_\omega \nabla p|^2 + p^2) \circ \Phi]\, r\, dr\, dt\, d\theta \ ,$$

and

$$\|p\|^2_{H^1(\Omega)} = \iiint_{T^\star_\alpha} |\mathbb{A}_\omega \nabla(\Psi(p))|^2 + (\Psi(p))^2\, dx.$$

This means that $\|p\|^2_{H^1(\Omega)} = \|\Psi(p)\|^2_{H^1_\Omega}$. The application Ψ is isometric from $\mathcal{F}_\alpha$ to $\mathcal{F}^\star_\alpha$. Equality $\mathcal{E}_\varepsilon(p) = \mathcal{E}^\star_\varepsilon(\Psi(p))$ can be proved with the same arguments. $\square$

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Chapitre 2

Un modèle de Mumford-Shah anisotrope et binaire

2.1 Résumé

Dans ce deuxième chapitre, nous modifions l'énergie de Mumford-Shah afin d'inclure la géométrie du problème *à l'intérieur* du modèle. Nous conservons toutefois l'hypothèse simplificatrice de binarité de l'image afin de faire abstraction des inhomogénéités de luminosité et pour nous concentrer (pour l'instant) sur les aspects géométriques.

Puisque nous maintenons l'hypothèse de binarité, nous allons assimiler le bruit à de petits ensembles dont le diamètre est comparable à la section α des tubes les plus fins. Nous souhaitons que α soit un seuil critique de détection, c'est-à-dire que le processus de segmentation que nous souhaitons construire doit vérifier les deux contraintes suivantes.

i) Tout ensemble de diamètre inférieur à α ne doit pas être détecté.

ii) Tout tube $T_{\ell,\alpha}$ de longueur ℓ et de section α tels que $\alpha \ll \ell$ doit être détecté.

Soit B_α une boule de rayon α, disjointe de $T_{\ell,\alpha}$, et $g = \mathbf{1}_{B_\alpha} + \mathbf{1}_{T_{\ell,\alpha}}$.

Notons $\mathcal{E}$ une énergie définie sur les fonctions binaires, dont la solution du problème de minimisation satisfait les deux conditions précédentes. En particulier, cette énergie doit vérifier :

i) $\mathcal{E}(\mathbf{1}_{T_{\ell,\alpha}}) < \mathcal{E}(g)$,

ii) $\mathcal{E}(g) < \mathcal{E}(\mathbf{1}_{B_\alpha})$.

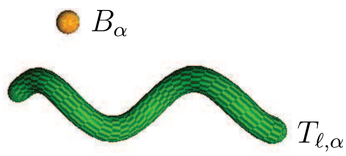

FIGURE 2.1.1 – Decomposition de g

Supposons d'abord que $\mathcal{E}$ est l'énergie de Mumford-Shah binaire étudiée dans le premier chapitre. En considérant les équivalences suivantes

$$\begin{array}{cc} \mathrm{Vol}^n(B_\alpha) \sim \alpha^n & \mathrm{Vol}^{n-1}(\partial B_\alpha) \sim \alpha^{n-1} \\ \mathrm{Vol}^n(T_{\ell,\alpha}) \sim \ell\alpha^{n-1} & \mathrm{Vol}^{n-1}(\partial T_{\ell,\alpha}) \sim \ell\alpha^{n-2} \end{array}$$

la condition $i)$ est équivalente à $\alpha < \beta$ alors que la condition $ii)$ est équivalente à $\beta < \alpha$. L'énergie de Mumford-Shah n'est donc pas adaptée à notre problème. Nous devons plutôt introduire une énergie qui privilégie les ensembles admettant une élongation suivant une direction. Pour cela, nous introduisons une nouvelle inconnue, $\mathbf{c} : \Omega \longrightarrow \mathbb{S}^{n-1}$ comme un champ de vecteurs de norme 1. Nous appellerons l'*action* de $\mathbf{c}$ sur un ensemble A la quantité suivante

$$\mathrm{Action}(A, \mathbf{c}) = \int_{\partial A} |\mathbf{c} \cdot \nu_A| \mathrm{dVol}^{n-1},$$

où ν_A est un vecteur normal à ∂A. Afin de forcer le champ $\mathbf{c}$ à ne pas admettre de discontinuités, nous supposons qu'il minimise le terme de régularisation suivant

$$\mathrm{Reg}(\mathbf{c}) = \int_\Omega \|D\mathbf{c}\|^r \mathrm{dVol}^n,$$

où $r > n$ assure que le champ est régulier (Dc représentant le champ dérivé). Pour $\ell \ll \alpha$, un champ qui minimise la somme de son action sur $T_{\ell,\alpha}$ et du terme régularisation doit être tangent au tube le long de sa longueur.

Nous introduisons alors l'énergie suivante

$$\mathcal{E}(\mathbf{1}_A, \mathbf{c}) = \int_\Omega (\mathbf{1}_A - g)^2 \mathrm{d}x + \beta \underbrace{\left(\mathcal{H}^{n-1}(\partial A) + \mu \mathrm{Action}(\partial A, \mathbf{c})\right)}_{\text{terme anisotrope}} + \gamma \mathrm{Reg}(\mathbf{c}), \tag{2.1.1}$$

où β, μ et γ sont des paramètres strictement positifs. Ainsi, la condition $i)$ est équivalente à

$$\alpha^n < \beta(\alpha^{n-1} + \mu\alpha^{n-1}), \tag{2.1.2}$$

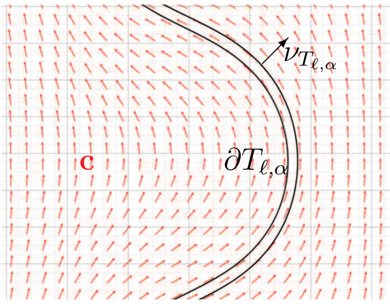

FIGURE 2.1.2 – Un champ $\mathbf{c}$ tangent à $T_{\ell,\alpha}$

alors que la condition *ii*) est équivalente à

$$\beta(\ell\alpha^{n-2} + \mu\alpha^{n-1}) < \ell\alpha^{n-1}. \tag{2.1.3}$$

Les deux conditions (2.1.3) et (2.1.2) ne sont plus en contradiction pour α petit et $\alpha \ll \ell$. Pour s'en convaincre, nous vérifions qu'elles sont satisfaites en prenant

$$\beta = \frac{\alpha}{2}, \quad \mu < \frac{\ell}{\alpha}, \quad \gamma = \mu.$$

Nous verrons dans cette partie que le terme anisotrope $\mathcal{H}^{n-1}(\partial A) + \mu \mathrm{Action}(\partial A, \mathbf{c})$ dans (2.1.1) est équivalent à la mesure anisotrope $(n-1)$ dimensionnelle de ∂F donnée par

$$\int_{\partial A} \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}$$

où ν est un vecteur normal à ∂A et

$$\mathbf{M}(x) = \mathrm{Id}_n + \mu \mathbf{c}(x)(\mathbf{c}(x))^t$$

pour tout $x \in \Omega$. La nouvelle variable $\mathbf{M}$ est donc une application définie sur Ω à valeurs dans l'ensemble des matrices symétriques définies positives. Pour tout x sur le tube, la matrice $\mathbf{M}(x)$ possède une valeur propre principale *dans la direction* du tube. Pour une fonction indicatrice $p = \mathbf{1}_A$, nous introduisons alors l'énergie suivante

$$E(\mathbf{1}_A, \mathbf{M}) = \int_\Omega (\mathbf{1}_A - g)^2 dx + \beta \int_{\partial A} \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + \gamma \int_\Omega \|D\mathbf{M}\|^r \mathrm{d}x,$$

où ν est un vecteur normal à ∂A. Afin de montrer que ce problème admet une solution, nous introduisons une formulation relaxée de cette énergie dans un espace fonctionnel adapté. Plus précisément, nous posons

$$E(p,\mathbf{M}) = \int_\Omega (p-g)^2 dx + \beta \int_{S_p} \langle \mathbf{M}\nu_p, \nu_p\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} + \gamma \int_\Omega \|D\mathbf{M}\|^r \mathrm{d}x,$$

où p est une fonction spéciale à variation bornée (SBV(Ω)(voir par exemple [AFP00]) prenant ses valeurs sur $\{0;1\}$ et S_p est son ensemble des sauts. Nous montrerons dans cette partie que cette énergie admet un minimimum dans cet espace de fonctions (Théorème 4.1). Afin de disposer d'une énergie ne dépendant que d'une intégration par rapport à la mesure de Lebesgue, nous posons

$$E_\varepsilon(p,\mathbf{M}) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_\Omega \left(9\varepsilon \langle \mathbf{M}\nabla p, \nabla p\rangle + \frac{p^2(1-p)^2}{\varepsilon} \right) \mathrm{d}x + \gamma \int_\Omega \|D\mathbf{M}\|^r \mathrm{d}x.$$

et nous montrons que $(E_\varepsilon)_\varepsilon$ Γ-converge vers E pour $\varepsilon \to 0^+$ (Théorème 5.1.).

L'article [Vic15a] qui suit est accepté pour publication dans *Advances in Calculus of Variation.*

2.2 Article [Vic15a]

An Anisotropic Bimodal Energy for the Segmentation of thin tubes ans its approximation with Γ-convergence

David Vicente

July 4, 2015

Abstract

This work is a contribution to the problem of detection of thin structures, namely *tubes*, in a 2D or 3D image. We introduce a variational *bimodal* model for the case where the histogram of the image has two main modes. This model involves an energy functional that depends on a function and a Riemannian metric. One of the terms of this energy is the anisotropic perimeter associated to the dual metric. We perform an approximation of this functional and prove that it Γ-converges to the original one.

1 Introduction

For the study of some diseases, it is interesting to focus on the blood status in a vessel network, especially on the volume of its microvasculature. To assess this, *in vivo* mice brain angiography is performed. This is based on the injection of a contrast medium and an MRI imaging process.

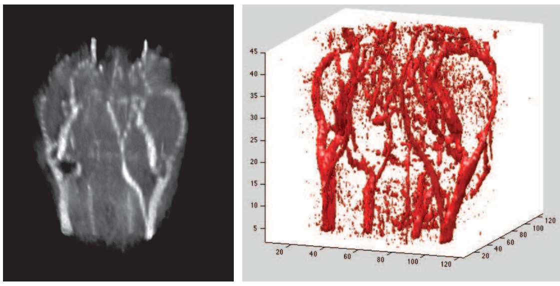

Figure 1.1: Mouse brain angiography and thresholding

The image we obtain is quite noisy (see figure 1.1). Performing a roughly manual thresholding gives a binary image where the smallest vessels have the same diameter scale as the noise. In the present work we focus on the thresholded image assuming it is binary. In order to remove the noise without deleting the thin tubes, we introduce an energy which distinguishes sets having a tubular geometry from others.

Let n be the dimension and $\Omega \subset \mathbb{R}^n$ be a domain. Let $g : \Omega \to [0;1]$ be an image with two modes 0 and 1. The analysis will consist in searching for a pair $(p, \mathbf{M})$, where $p : \Omega \to \{0;1\}$ and where $\mathbf{M} : \Omega \to \mathcal{S}_n(\mathbb{R})$ is a field of symmetric matrices. We assume that $p \in \mathrm{BV}(\Omega)$ and $\mathbf{M} \in W^{1,r}(\Omega; \mathcal{S}_n(\mathbb{R}))$, where the exponent r will be made precise later. The pair $(p, \mathbf{M})$ must minimize the functional

$$E(p, \mathbf{M}) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \gamma \|\mathbf{M}\|_{W^{1,r}(\Omega)},$$

where S_p is the jump set of p, $\boldsymbol{\nu}_p$ is a normal unit vector to S_p and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. The parameters β and γ are weights to be determined. In order to detect thin tubes, $\mathbf{M}$ must have the form

$$\mathbf{M} = \mathrm{Id}_n + \mu^2 \mathbf{c}\mathbf{c}^t,$$

where $\mathbf{c} : \Omega \to \mathbb{R}^n$ is an unknown unit vector field and μ is a fixed parameter. In particular, the matrix $\mathbf{M}(x)$ is symmetric definite positive for any $x \in \Omega$. The relationship between μ and the thickness of the tubes will be discussed later.

In [1], it is proved that the second term of the functional E is the anisotropic perimeter associated to the Riemannian metric $(x, \mathbf{v}) \in \Omega \times \mathbb{R}^n \to \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle^{1/2}$. If $\mathbf{M}$ is fixed, this functional inherits a lower semi-continuity property and it can be approximated in the sense of Γ-convergence by an adapted family of functionals. We generalize this work to the case where $\mathbf{M}$ is also an unknown (and depends on $x \in \Omega$). More precisely, we introduce the functional

$$E_\varepsilon(p, \mathbf{M}) = \int_\Omega (p - g)^2 \mathrm{d}x + \beta \int_\Omega \left(9\varepsilon \langle \mathbf{M}\nabla p, \nabla p\rangle + \frac{p^2(1-p)^2}{\varepsilon} \right) \mathrm{d}x + \gamma \|\mathbf{M}\|_{W^{1,r}(\Omega)},$$

where $p : \Omega \to [0; 1]$ is a regular function. We will prove that $(E_\varepsilon)_{\varepsilon>0}$ is an approximation of E when ε converges to 0^+.

In section 2, we set up the problem and give a geometric interpretation for the parameters β, γ, μ. In section 3, we recall some classical results and introduce the functional framework. Section 4 is devoted to the existence of solutions for the minimizing problem. Finally, in section 5, we introduce the approximation process and prove our main result, that is, the family $(E_\varepsilon)_{\varepsilon>0}$ Γ-converges to E.

2 Presentation of the model

In what follows, n represents the spatial dimension of the image with $n = 2$ or $n = 3$. We adopt the notation that

- $\langle \cdot, \cdot \rangle$ denotes the usual scalar product and $|\cdot|$ denotes the Euclidean norm of $\mathbb{R}^n$,
- bold symbols $\mathbf{v}, \mathbf{c}$ are reserved for vector or vector-valued functions and bold capital $\mathbf{M}$ for matrices or matrix-valued functions,
- $\mathbb{S}^{n-1}$ denotes the unit sphere of $\mathbb{R}^n$,
- $B_r(x)$ denotes a ball in $\mathbb{R}^n$ with center x and radius $r \geq 0$,
- $\|A\|$ denotes a generic norm in the space of $n \times n$ matrices,
- $\mathrm{sp}(A)$ denotes the eigenvalues of A counted with their multiplicities,
- $\mathcal{L}^n$ denotes the Lebesgue measure in $\mathbb{R}^n$ and $\mathcal{H}^{n-1}$ denotes the $(n-1)$-dimensional Hausdorff measure,
- $\mathcal{H}^{n-1} \llcorner A$ denotes the restriction of the Hausdorff measure to the set A.

2.1 An isotropic model

We give a heuristic way to introduce and motivate the model. We first present an isotropic model and show that it is not suitable for our problem. Then, we introduce an anisotropic term.

Let $\Omega \subset \mathbb{R}^n$ be the domain of the image. Formally, we say that $T_{\ell,\alpha}$ is a tube with section α and length ℓ if there exists a curve Γ such that $T_{\ell,\alpha}$ is the set of the points of Ω at distance less than α from Γ.

We consider the following segmentation problem. Let us give $\alpha > 0$ a critical level of detection: if a set has diameter less than α, then it is considered as noise and has to be removed. On the other hand, any tube $T_{\ell,\alpha}$ with section α negligible compared with its length ℓ has to be detected. Although the section of $T_{\ell,\alpha}$ is critical we want to detect it because of its specific geometry. A ball B_α with radius α is considered as noise not only because it has critical diameter but also because it does not fit the appropriate geometry of tubes.

Let $I \subset \Omega$ be a generic set and assume that we have the disjoint decomposition $I = T_{\ell,\alpha} \cup B_\alpha$. The segmentation problem consists in combining two constraints. The first one is to remove B_α-type sets, because they have small radius and no tubular geometry. The second one is to detect the tubes $T_{\ell,\alpha}$.

To this end, we consider an energy functional $\mathcal{E}$ defined on the sets of Ω. We say that a set F is a *better segmentation* than the set G if $\mathcal{E}(F) < \mathcal{E}(G)$. Here, the functional $\mathcal{E}$ is *adapted* to the problem if it satisfies the following conditions:

i) $\mathcal{E}(I \setminus B_\alpha) < \mathcal{E}(I)$,

ii) $\mathcal{E}(I) < \mathcal{E}(I \setminus T_{\ell,\alpha})$.

As a first step, we consider a "naive" energy defined on sets by

$$\mathcal{E}(F) = \mathrm{Vol}^n(F \vartriangle I) + \beta \mathrm{Vol}^{n-1}(\partial F),$$

where Vol^n and Vol^{n-1} are respectively the volume measures with dimension n and $n-1$, $F \vartriangle I$ is the symmetric difference of the sets and ∂F is the boundary of a set F. This model favors the detection of sets which minimize the ratio $\mathrm{Vol}^{n-1}/\mathrm{Vol}^n$, so it can not satisfy the constraints we imposed: if $T_{\ell,\alpha}$ is a linear tube then condition (i) leads to $\alpha < \beta$ and condition (ii) leads to $\beta < \alpha$. In order to overcome this contradiction, we will introduce a new model.

2.2 An anisotropic model

Let us introduce an energy term that favors anisotropic sets. Let $\mathbf{c} : \Omega \to \mathbb{S}^{n-1}$ be an unknown and unit vector field that represents a direction at each point of the image. Let $x \in \partial F$ and $\boldsymbol{\nu}_F(x)$ be a unit normal vector to the surface ∂F at x and set

$$\mathcal{N}(\partial F, \mathbf{c}) = \int_{\partial F} |\langle \mathbf{c}, \boldsymbol{\nu}_F \rangle| \mathrm{dVol}^{n-1}.$$

As $\alpha \ll \ell$, a field which minimizes $\mathcal{N}(\partial T_{\ell,\alpha}, \mathbf{c})$ should be tangent to $\partial T_{\ell,\alpha}$. Moreover, we introduce a regularization term defined on the vector field as

$$\mathrm{Reg}(\mathbf{c}) = \int_{\Omega} \|D\mathbf{c}\|^r \mathrm{dVol}^n,$$

where $\|\cdot\|$ is a pointwise matrix norm and we fix $r > n$ (not necessary an integer) to ensure that the field is regular. Indeed, if $r > n$ and $\int_\Omega \|D\mathbf{c}\|^r < \infty$ then $\mathbf{c}$ is continuous. The new expression of the energy is

$$\mathcal{E}(F, \mathbf{c}) = \mathrm{Vol}^n(F \vartriangle I) + \beta \underbrace{\left(\mathrm{Vol}^{n-1}(\partial F) + \mu \mathcal{N}(\partial F, \mathbf{c})\right)}_{\text{anisotropic term}} + \gamma \mathrm{Reg}(\mathbf{c}), \tag{2.1}$$

where β, μ and γ are positive weights. We still have to check the conditions (i) and (ii) of subsection 2.1. Assume that $T_{\ell,\alpha}$ is a linear and rigid tube of length ℓ and section α. Obviously, the best choice of $\mathbf{c}$ is in the direction of the tube. Using

$$\mathrm{Vol}^n(T_{\ell,\alpha}) \sim \ell \alpha^{n-1}, \quad \mathrm{Vol}^{n-1}(T_{\ell,\alpha}) \sim \ell \alpha^{n-2}, \quad \mathcal{N}(\partial T_{\ell,\alpha}, \mathbf{c}) \sim \alpha^{n-1},$$

condition (ii) is equivalent to

$$\beta(\ell \alpha^{n-2} + \mu \alpha^{n-1}) < \ell \alpha^{n-1}. \tag{2.2}$$

For a ball B_α, the homothetic change of variables between B_α and B_1, where we denote $\mathbf{c}_1 = \mathbf{c}(\alpha\,\cdot)$, gives:

$$\beta\mu\mathcal{N}(\partial B_\alpha, \mathbf{c}) + \gamma \mathrm{Reg}(\mathbf{c}) = \beta\mu\alpha^{n-1}\mathcal{N}(\partial B_1, \mathbf{c}_1) + \gamma \alpha^{n-r} \mathrm{Reg}(\mathbf{c}_1).$$

As α is small and $r > n$, if $\mu \sim \gamma$ then the parameter $\mu\alpha^{n-1}$ is negligible with respect to $\gamma\alpha^{n-r}$. As a conclusion, the regularization is more important than the normal term for balls with small radius α. The best choice for $\mathbf{c}$ is a constant field. In this case, condition (i) is equivalent to

$$\alpha^n < \beta(\alpha^{n-1} + \mu\alpha^{n-1}). \tag{2.3}$$

The two conditions (2.2) and (2.3) are now compatible when α is small and $\alpha \ll \ell$. For example, we may take

$$\beta = \frac{\alpha}{2}, \quad \mu < \frac{\ell}{\alpha}, \quad \gamma = \mu.$$

2.3 Functional formulation

In the sequel, we will formulate (2.1) as a minimization problem for functions by relating sets and functions via indicator functions. We define an image as a function $g : \Omega \to [0; 1]$. We assume that the domain $\Omega \subset \mathbb{R}^n$ is Lipschitz regular. Our fondamental assumption is that the histogram distribution of the image contains two main modes that we assume to be 0 and 1. Roughly speaking, g is almost equal to an indicator function. The unknown is a pair $(p, \mathbf{c})$ where $p : \Omega \to \{0; 1\}$ is a binary function and $\mathbf{c} : \Omega \to \mathbb{S}^{n-1}$ is an unit vector field which minimizes the energy

$$\int_\Omega (p-g)^2 \mathrm{d}x + \beta \left(\mathcal{H}^{n-1}(S_p) + \mu \int_{S_p} |\langle \mathbf{c}, \boldsymbol{\nu}_p \rangle| \, \mathrm{d}\mathcal{H}^{n-1} \right) + \gamma \int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x, \tag{2.4}$$

where $\mathrm{d}x$ stands for to the n-dimensional Lebesgue measure, S_p is the jump set of p, $\boldsymbol{\nu}_p : S_p \to \mathbb{S}^{n-1}$ is a normal unit vector of S_p and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We can rewrite (2.4) as

$$\int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_{S_p} (1 + \mu |\langle \mathbf{c}, \boldsymbol{\nu}_p \rangle|) \ \mathrm{d}\mathcal{H}^{n-1} + \gamma \int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x. \tag{2.5}$$

The second term of the functional (2.5) corresponds to the anisotropic perimeter of S_p according to the metric $\phi : \Omega \times \mathbb{R}^n \to [0; +\infty[$ defined as

$$\phi(x, \mathbf{v}) = |\mathbf{v}| + \mu |\langle \mathbf{c}(x), \mathbf{v} \rangle|.$$

For more convenience in the calculus, we will adopt the equivalent quadratic form

$$\int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_{S_p} \sqrt{1 + \mu^2 \langle \mathbf{c}, \boldsymbol{\nu}_p \rangle^2} \ \mathrm{d}\mathcal{H}^{n-1} + \gamma \int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x,$$

which has an obvious invariance. Indeed, the functional (2.5) takes the same value for $\mathbf{c}$ and $-\mathbf{c}$. To symmetrize it, we replace the unknown vector field $\mathbf{c}$ by a field of matrices which takes the form $\mathbf{M} = \mathrm{Id}_n + \mu^2 \mathbf{c}\mathbf{c}^t$ so that

$$\langle \mathbf{M}(x)\mathbf{v}, \mathbf{v} \rangle = |\mathbf{v}|^2 + \mu^2 \ \langle \mathbf{c}(x), \mathbf{v} \rangle^2$$

for all $x \in \Omega$. We introduce the final version of the functional as

$$E(p, \mathbf{M}) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \gamma \|\mathbf{M}\|_{W^{1,r}(\Omega)}. \tag{2.6}$$

In [1], it is proved that the quantity

$$\int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$$

is the anisotropic perimeter associated to the dual metric associated to $\mathbf{M}$. We can explicitly calculate this metric as

$$(x, \mathbf{v}) \to |\mathbf{v}|^2 - \frac{\mu^2}{1+\mu^2} \langle \mathbf{c}(x), \mathbf{v} \rangle^2.$$

The unit ball for this metric is an elongated ellipsoid in the direction of $\mathbf{c}(x)$. The points in the direction of $\mathbf{c}(x)$ are closer to x than the points in the orthogonal directions and the ratio of the elongation is equal to $\sqrt{1+\mu^2}$.

3 Functional framework and basic tools

In the following, we assume that the parameters are fixed, that is, $\beta = \mu = \gamma = 1$. We denote by $W^{1,2}(\Omega; [0; 1])$ the set of functions p which belong to $W^{1,2}(\Omega)$ such that $p(x) \in [0; 1]$ for almost every $x \in \Omega$. Let $\mathcal{S}_n(\mathbb{R})$ be the space of $n \times n$ symmetric matrices and let $W^{1,r}(\Omega; \mathcal{S}_n(\mathbb{R}))$ be the associated Sobolev space. Let $\mathcal{G}$ be the subset of $\mathcal{S}_n(\mathbb{R})$ defined by

$$\mathcal{G} = \{\mathrm{Id}_n + \mathbf{c}\mathbf{c}^t : \quad \mathbf{c} \in \mathbb{S}^{n-1}\}.$$

Obviously, any matrix which belongs to $\mathcal{G}$ is symmetric definite positive. We introduce the space

$$W^{1,r}(\Omega;\mathcal{G}) = \left\{\mathbf{M} \in W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))\colon \text{ for all } x \in \Omega, \mathbf{M}(x) \in \mathcal{G}\right\}.$$

The real number r is determined according to the classical Sobolev embedding theorem which ensures that, if $r > n$, then the inclusion

$$W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R})) \hookrightarrow \mathcal{C}(\Omega;\mathcal{S}_n(\mathbb{R})) \tag{3.1}$$

is compact, where $\mathcal{C}(\Omega;\mathcal{S}_n(\mathbb{R}))$ is the space of continuous functions defined on Ω which take their values in $\mathcal{S}_n(\mathbb{R})$ endowed with the L^∞-norm. This result is the main argument in the proof of the following proposition.

Proposition 3.1. *If $r > n$, then $W^{1,r}(\Omega;\mathcal{G})$ is closed in $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$ for the weak topology associated to the Sobolev norm.*

To prove this, we need clause (ii) of the following lemma. Clause (i) will be useful throughout the paper.

Lemma 3.1. *For $M \in \mathcal{S}_n(\mathbb{R})$, we have that*

i) $\mathbf{M} \in \mathcal{G}$ *implies that for all* $\mathbf{v} \in \mathbb{R}^n$ *there holds* $|\mathbf{v}|^2 \leq \langle \mathbf{Mv}, \mathbf{v}\rangle \leq 2|\mathbf{v}|^2$,

ii) $\mathbf{M} \in \mathcal{G}$ *if and only if* $sp(\mathbf{M}) = \{1; 1; 2\}$.

Proof. If $\mathbf{M} \in \mathcal{G}$ then there exists $\mathbf{c} \in \mathbb{S}^{n-1}$ such that $\mathbf{M} = \mathrm{Id}_n + \mathbf{cc}^t$. So, we have

$$\langle \mathbf{Mv}, \mathbf{v}\rangle = |\mathbf{v}|^2 + \langle \mathbf{c}, \mathbf{v}\rangle^2 \quad \text{and} \quad |\mathbf{v}|^2 \leq \langle \mathbf{Mv}, \mathbf{v}\rangle \leq 2|\mathbf{v}|^2.$$

Moreover, $\mathbf{Mc} = 2\mathbf{c}$ and the restriction of $\mathbf{M}$ to $\mathrm{Vect}(\mathbf{c})^\perp$ is the identity, so $\mathrm{sp}(\mathbf{M}) = \{1; 1; 2\}$.

Conversely, if $\mathrm{sp}(\mathbf{M}) = \{1; 1; 2\}$, we let $\mathbf{c}$ be an unit eigenvector associated to the eigenvalue 2. Since $\mathbf{M}$ is symmetric, it follows that $\mathrm{Vect}(\mathbf{c})^\perp$ is stable by $\mathbf{M}$ and its restriction to $\mathrm{Vect}(\mathbf{c})^\perp$ is the identity. As $\mathbf{M}$ and $\mathrm{Id}_n +\ \mathbf{cc}^t$ coincide in $\mathrm{Vect}(\mathbf{c})^\perp$ and $\mathrm{Vect}(\mathbf{c})$, they are equal. □

Now, we can prove the Proposition 3.1.

Proof. Let $(\mathbf{M}_k)_k \subset W^{1,r}(\Omega;\mathcal{G})$ be a Cauchy sequence for the weak topology associated to $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$. Since $r > n$, the inclusion

$$W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R})) \subset \mathcal{C}(\Omega;\mathcal{S}_n(\mathbb{R}))$$

is compact. So, $(\mathbf{M}_k)_k$ is also a Cauchy sequence for the $L^\infty(\Omega; S_n(\mathbb{R}))$ norm and, for $x \in \Omega$ fixed, $(\mathbf{M}_k(x))_k$ converges to a matrix $\mathbf{M}(x)$. Since the two characterizations of Lemma 3.1 are stable under the limit, it follows that $\mathbf{M}(x)$ verifies this two conditions. This proves that $W^{1,r}(\Omega;\mathcal{G})$ is closed in $W^{1,r}(\Omega; S_n(\mathbb{R}))$.

□

We need a density result for smooth functions as well.

Proposition 3.2. *The space $\mathcal{C}^\infty \cap W^{1,r}(\Omega;\mathcal{G})$ is dense in $W^{1,r}(\Omega;\mathcal{G})$ for the strong topology of $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$.*

Proof. It is well known that maps of Sobolev spaces $W^{1,r}$ between two compact manifolds can be approximated by smooth maps in the case $r > n$ (see [2] for example). So, it suffices to prove that $\mathcal{G}$ is a compact $\mathcal{C}^\infty$-submanifold of $\mathcal{S}_n(\mathbb{R})$. To this end, we consider

$$\Psi : \mathbb{R}^n \setminus \{0\} \to \mathcal{S}_n(\mathbb{R}), \quad \mathbf{c} \mapsto \mathrm{Id}_n + \mathbf{cc}^t.$$

For any $\mathbf{c} \in \mathbb{R}^n \setminus \{0\}$, Ψ is differentiable and $D\Psi_{(\mathbf{c})}(\mathbf{h}) = \mathbf{ch}^t + \mathbf{hc}^t$. As the function $\mathbf{c} \mapsto D\Psi_{(\mathbf{c})}$ is linear, it follows that Ψ is a $\mathcal{C}^\infty$-function. If $\mathbf{h} \in \ker(D\Psi_{(\mathbf{c})})$, then $\mathbf{ch}^t + \mathbf{hc}^t$ is the null matrix and in particular there holds

$$0 = \mathbf{c}^t(\mathbf{ch}^t + \mathbf{hc}^t)\mathbf{c} = 2\langle \mathbf{c}, \mathbf{h}\rangle, \quad 0 = \mathbf{h}^t(\mathbf{ch}^t + \mathbf{hc}^t)\mathbf{c} = \langle \mathbf{c}, \mathbf{h}\rangle + |\mathbf{h}|^2.$$

This gives $\mathbf{h} = 0$ and, as a result, Ψ is a $\mathcal{C}^\infty$-immersion. Since $\mathbb{S}^{n-1}$ is a compact $\mathcal{C}^\infty$-submanifold of $\mathbb{R}^n \setminus \{0\}$ and $\mathcal{G} = \Psi(\mathbb{S}^{n-1})$, it follows that $\mathcal{G}$ is a compact $\mathcal{C}^\infty$-submanifold of $\mathcal{S}_n(\mathbb{R})$. □

Let $\mathcal{C}_c(\Omega;\mathbb{R}^n)$ be the space of continuous functions with compact support in Ω and with values in $\mathbb{R}^n$. We denote by $\mathcal{C}_0(\Omega;\mathbb{R}^n)$ the closure of $\mathcal{C}_c(\Omega;\mathbb{R}^n)$. Let $\mathcal{M}(\Omega)$ be the space of Radon measures and $\mathcal{M}(\Omega;\mathbb{R}^n)$ be the space of vectorial Radon measures over Ω. The space $\mathcal{M}(\Omega;\mathbb{R}^n)$, endowed with the norm

$$\|\lambda\|_{\mathcal{M}(\Omega;\mathbb{R}^n)} = \sup\left\{\int_\Omega \varphi\cdot \mathrm{d}\lambda : \varphi\in\mathcal{C}_c(\Omega;\mathbb{R}^n), \|\varphi\|_{L^\infty}\leq 1\right\}$$

is a Banach space. As this topology is quite restrictive in our case, we introduce a weaker topology.

Definition 3.1. *For $\lambda\in\mathcal{M}(\Omega;\mathbb{R}^n)$, the sequence $(\lambda_k)_k\subset\mathcal{M}(\Omega;\mathbb{R}^n)$ weakly* converges to λ if*

$$\lim_k \int_\Omega \varphi\cdot d\lambda_k = \int_\Omega \varphi\cdot d\lambda$$

for every $\varphi\in\mathcal{C}_0(\Omega;\mathbb{R}^n)$.

Endowed of this topology, the space $\mathcal{M}(\Omega;\mathbb{R}^n)$ satisfies a compactness property.

Theorem 3.1. *If $(\lambda_k)_k\subset\mathcal{M}(\Omega;\mathbb{R}^n)$ is a bounded sequence for the topology of the norm, then it has a weakly* converging subsequence. Moreover, the norm is lower semicontinuous with respect to the weak* convergence.*

Now, let $\varphi:\Omega\times\mathbb{R}^n\to\mathbb{R}^+$ be a sublinear function with respect to the second variable, that is,

i) for all $(x,\mathbf{v}_1,\mathbf{v}_2)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^n$ there holds $\varphi(x,\mathbf{v}_1+\mathbf{v}_2)\leq\varphi(x,\mathbf{v}_1)+\varphi(x,\mathbf{v}_2)$,

ii) for all $(x,\mathbf{v},t)\in\Omega\times\mathbb{R}^n\times\mathbb{R}^+$ there holds $\varphi(x,t\mathbf{v})=t\varphi(x,\mathbf{v})$.

Suppose that θ is a positive Radon measure and λ is a vectorial Radon measure on Ω. According to Besicovitch derivation theorem (see [3]), the limit

$$\lim_{r\to 0}\frac{\lambda(B_r(x))}{\theta(B_r(x))}$$

exists and is finite for θ almost every x and we denote this limit by $(\mathrm{d}\lambda/\mathrm{d}\theta)(x)$ when it exists. We recall that λ is absolutely continuous with respect to θ if $\lambda(A)=0$ whenever $\theta(A)=0$. When this holds, we write $\lambda\ll\theta$. We consider the convex functional defined on the space $\mathcal{M}(\Omega;\mathbb{R}^n)$ by

$$\Phi:\lambda\in\mathcal{M}(\Omega;\mathbb{R}^n)\mapsto\int_\Omega\varphi\left(x,\frac{\mathrm{d}\lambda}{\mathrm{d}\theta}\right)\mathrm{d}\theta, \tag{3.2}$$

where θ is a positive measure such that $\lambda\ll\theta$. It is shown in [4] that the integral in (3.2) does not depend on the choice of θ. For that reason, we will write it in the condensed form

$$\Phi(\lambda)=\int_\Omega\varphi(x,\lambda).$$

The functional Φ has the following continuity properties which are proved in [5].

Proposition 3.3. *The following statements hold*

i) If φ is a lower semicontinuous on $\Omega\times\mathbb{R}^n$, then Φ is lower semicontinuous on $\mathcal{M}(\Omega;\mathbb{R}^n)$ for the topology introduced in Definition 3.1.

ii) Assume that φ is continuous on $\Omega\times\mathbb{R}^n$. If $(\lambda_k)_k$ weakly converges to λ and if, moreover, $\int_\Omega|\lambda_k|\to\int_\Omega|\lambda|$, then $\Phi(\lambda_k)$ converges to $\Phi(\lambda)$.*

A function $u\in L^1(\Omega)$ is said to have bounded variation if

$$\sup\left\{\int_\Omega u\mathrm{div}(\varphi)\colon\varphi\in\mathcal{C}_0^1(\Omega;\mathbb{R}^n), \|\varphi\|_{L^\infty}\leq 1\right\}<+\infty.$$

We denote by $\mathrm{TV}(u)$ this upper bound and the set of such functions by $\mathrm{BV}(\Omega)$. A measurable set $A\subset\Omega$ is said to have finite perimeter if $\mathbf{1}_A\in\mathrm{BV}(\Omega)$.

The space $\mathrm{BV}(\Omega)$, equipped with the following norm

$$\|u\|_{BV(\Omega)}=\|u\|_{L^1(\Omega)}+TV(u)$$

is a Banach space. According to the Riesz representation theorem, if $u\in BV(\Omega)$ then its derivative, in the sense of distributions, belongs to $\mathcal{M}(\Omega;\mathbb{R}^n)$ and we denote it by Du. The topology of the norm in $BV(\Omega)$ is quite restrictive in our case, so we consider a weaker one.

Definition 3.2. *A sequence $(u_k)_k \subset BV(\Omega)$ weakly* converges to $u \in BV(\Omega)$ if $(u_k)_k$ converges to u in $L^1(\Omega)$ and Du_k weakly* converges to Du in $\mathcal{M}(\Omega;\mathbb{R}^n)$.*

Several authors simply call it the *weak* convergence. To avoid confusion with the usual topology associated to the dual space of $BV(\Omega)$, we prefer to call it here the *weak** convergence. The space $BV(\Omega)$ satisfies a compactness result.

Theorem 3.2. *If $(u_k)_k \subset BV(\Omega)$ is such that $(\|u_k\|_{BV(\Omega)})$ is bounded sequence, then it has a weakly* converging subsequence.*

We give a variant of the coarea formula extended to the sublinear functionals (see [6]).

Proposition 3.4. *Let $\Phi(x,s,\mathbf{v})$ be a Borel function of $\Omega\times\mathbb{R}\times\mathbb{R}^n$ which is sublinear in $\mathbf{v}$. Let p be a Lipschitz continuous function on Ω and for $t>0$ let $S_t=\{x\in\Omega; p(x)<t\}$. Then, for almost all $t\in\mathbb{R}$, S_t has finite perimeter in Ω and we have*

$$\int_\Omega \Phi(x,p,Dp)\,dx = \int_{\mathbb{R}} dt \int_\Omega \Phi(x,t,D\mathbf{1}_{S_t}).$$

We finish with some classical BV-geometric properties we need in the sequel (see [7]).

Definition 3.3. *Let $A\subset\Omega$ be a measurable set. A point $x\in\Omega$ belongs to the* ***measure theoretic boundary*** *of A if*

$$\limsup_{r\to0^+}\frac{\mathcal{L}^n(B(x,r)\cap A)}{r^n}>0 \quad \textit{and} \quad \liminf_{r\to0^+}\frac{\mathcal{L}^n(B(x,r)\setminus A)}{r^n}>0.$$

We denote by ∂^*A the measure theoretic boundary of A .

Theorem 3.3. *For a set $A\subset\Omega$ with finite perimeter, the following generalized Gauss-Green formula holds. For $\mathcal{H}^{n-1}$ almost every $x\in\partial^*A$, there exists a vector $\boldsymbol{\nu}(x)\in\mathbb{S}^{n-1}$, called the* ***inner normal vector*** *to A at x, such that*

$$\int_\Omega \mathbf{1}_A\, div(\varphi)\,dx = -\int_{\partial*A\cap\Omega}\varphi\cdot\boldsymbol{\nu}\,d\mathcal{H}^{n-1} \quad \textit{for all } \varphi\in\mathcal{C}^1_c(\Omega;\mathbb{R}^n),$$

that is, $D\mathbf{1}_A=\boldsymbol{\nu}\mathcal{H}^{n-1}\llcorner\partial^*A\cap\Omega$.

Theorem 3.4. *Let $A\subset\Omega$ be a set with finite perimeter. There exists a pairwise disjoint family of sets $(S_i)_i$ and a set $N\subset\Omega$ such that*

i) S_i *is a* $\mathcal{C}^1$ *and compact hypersurface of* Ω *for all* i,

ii) $\mathcal{H}^{n-1}(N)=0$,

iii) $\partial^*A = N\cup(\bigcup_i S_i)$.

Definition 3.4. *Let $A\subset\Omega$ be a set with finite perimeter and $p=\mathbf{1}_A$. From Theorems 3.4 and 3.3 we get that $S_p=\partial^*A$ and $\boldsymbol{\nu}_p=\boldsymbol{\nu}$.*

Proposition 3.5. *Let $A\subset\Omega$ be a set with finite perimeter and $p=\mathbf{1}_A$. Then, we have $\|Dp\|_{\mathcal{M}}=\mathcal{H}^{n-1}(S_p)$.*

The following lemma is proved in [8]. It asserts that every set with finite perimeter can be approximated by a sequence of smooth subsets of $\mathbb{R}^n$, all having the same volume inside Ω and each of these boundaries satisfies a measure theoretic transversality condition with respect to Ω.

Lemma 3.2. *Let Ω be an open and bounded subset of $\mathbb{R}^n$ with Lipschitz continuous boundary, and let A be a measurable subset of Ω. If A and $\Omega\setminus A$ both contain a non-empty open ball, then there exists a sequence $(A_k)_k$ of open bounded subsets of $\mathbb{R}^n$ with smooth boundaries such that*

i) $\lim_{k\to\infty}\mathcal{L}^n((A_k\cap\Omega)\,\triangle\, A)=0$ *and* $\lim_{k\to\infty}\mathcal{H}^{n-1}(\partial A_k)=TV(\mathbf{1}_A)$;

ii) $\mathcal{L}^n(A_k\cap\Omega)=\mathcal{L}^n(A)$ *for* k *large enough;*

iii) $\mathcal{H}^{n-1}(\partial A_k\cap\partial\Omega)=0$ *for* k *large enough.*

4 Existence of solutions

In this section we prove that the function defined in (2.6) admits at least one minimizer. Let us first introduce the appropriate functional spaces and their associated topologies.

- $\mathbb{B}(\Omega;[0;1]) = \{p:\Omega\to\mathbb{R} \text{ measurable }: p(x)\in[0;1] \text{ a.e. } x\in\Omega\}$ endowed with the almost everywhere convergence topology.
- $W^{1,r}(\Omega;\mathcal{G})$ endowed with the weak topology associated to the Sobolev norm $\|\cdot\|_{W^{1,r}(\Omega)}$.
- $\mathcal{X} = \mathbb{B}(\Omega;[0;1])\times W^{1,r}(\Omega;\mathcal{G})$ endowed with the product of the topologies. For a sequence $(p_k,\mathbf{M}_k)_k$ which converges to $(p,\mathbf{M})$ for this topology, we write $(p_k,\mathbf{M}_k)\xrightarrow{\mathcal{T}}(p,\mathbf{M})$. Since these spaces are metrizable, it follows that $(\mathcal{X},\mathcal{T})$ is also metrizable.
- $\mathrm{BV}(\Omega;\{0;1\}) = \{p\in\mathrm{BV}(\Omega)\colon\ p(x)\in\{0;1\}\ \text{ a.e. } x\in\Omega\}$;
- $\mathcal{Y} = \mathrm{BV}(\Omega;\{0;1\})\times W^{1,r}(\Omega;\mathcal{G})$.

We recall that

$$E(p,\mathbf{M}) = \begin{cases} \displaystyle\int_\Omega (p-g)^2\mathrm{d}x + \int_{S_p}\langle\mathbf{M}\boldsymbol{\nu}_p,\boldsymbol{\nu}_p\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1} + \|\mathbf{M}\|_{W^{1,r}(\Omega)} & \text{for } (p,\mathbf{M})\in\mathcal{Y},\\ +\infty & \text{for } (p,\mathbf{M})\in\mathcal{X}\setminus\mathcal{Y}.\end{cases}$$

We consider the minimization problem

$$(\mathcal{P}):\qquad \mathrm{Min}\,\{E(p,\mathbf{M})\colon (p,\mathbf{M})\in\mathcal{X}\}\,. \tag{4.1}$$

We prove that problem $(\mathcal{P})$ admits at least one solution by using the direct method of the calculus of variations.

Proposition 4.1 (Compactness)**.** *Let $(p_k,\mathbf{M}_k)_k\subset\mathcal{X}$ such that $(E(p_k,\mathbf{M}_k))_k$ is bounded. Then, there exists a subsequence, still denoted by $(p_k,\mathbf{M}_k)_k$, and $(p,\mathbf{M})\in\mathcal{Y}$ such that*

$$(p_k,\mathbf{M}_k)\xrightarrow{\mathcal{T}}(p,\mathbf{M}).$$

Proof. As $E(p_k,\mathbf{M}_k)$ is finite for any k, we have $(p_k,\mathbf{M}_k)_k\subset\mathcal{Y}$. We separate the arguments of the proof for $(p_k)_k$ and $(\mathbf{M}_k)_k$.

Step 1: Compactness result for $(p_k)_k$. Since p_k takes its values in $[0;1]$ and Ω is bounded, it follows that $(p_k)_k$ is a bounded sequence of $L^1(\Omega)$. According to Lemma 3.1, we have

$$1\leq\langle\mathbf{M}_k\boldsymbol{\nu}_{p_k},\boldsymbol{\nu}_{p_k}\rangle.$$

Integration with respect to $\mathcal{H}^{n-1}\llcorner S_{p_k}$ gives $\mathcal{H}^{n-1}(S_{p_k})\leq E(p_k,\mathbf{M}_k)$. According to Proposition 3.5, we have $\|Dp_k\|_{\mathcal{M}} = \mathcal{H}^{n-1}(S_{p_k})$, so $(p_k)_k$ is a bounded sequence of $\mathrm{BV}(\Omega)$. According to Theorem 3.2, there exists a subsequence, still denoted by $(p_k)_k$, and some $p\in\mathrm{BV}(\Omega)$ such that $(p_k)_k$ weakly* converges to p. According to Theorem 3.2, $(p_k)_k$ converges to p for the $L^1(\Omega)$ norm. As p_k takes its values in $\{0;1\}$, we deduce that p takes its values in $\{0;1\}$.

Step 2: Compactness result for $(\mathbf{M}_k)_k$. As $\|\mathbf{M}_k\|_{W^{1,r}(\Omega)}\leq E(p_k,\mathbf{M}_k)$, it follows that the sequence $(\mathbf{M}_k)_k$ is bounded in $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$. According to the Banach-Alaoglu theorem, there exists a subsequence, still denoted by $(\mathbf{M}_k)_k$, and $\mathbf{M}\in W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$ such that $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M}$ in $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$. Then, according to Proposition 3.1, we have $\mathbf{M}\in W^{1,r}(\Omega;\mathcal{G})$. ☐

Proposition 4.2 (Semicontinuity)**.** *The functional $E:\mathcal{X}\to\mathbb{R}$ is lower semicontinuous for the topology $\mathcal{T}$.*

Proof. The lower semicontinuity of $p\to\int_\Omega(p-g)^2\mathrm{d}x$ and $\mathbf{M}\to\|\mathbf{M}\|_{W^{1,r}(\Omega)}$ are due to the lower semi-continuity of the norm, in $L^2(\Omega)$ and $W^{1,r}(\Omega)$, with respect to the weak topology. The remaining part of this result is the lower semicontinuity of

$$(p,\mathbf{M})\to\int_{S_p}\langle\mathbf{M}\boldsymbol{\nu}_p,\boldsymbol{\nu}_p\rangle^{1/2}\mathrm{d}\mathcal{H}^{n-1}.$$

We first prove the result with $\mathbf{M}$ fixed.

Step 1. Let $\mathbf{M} \in W^{1,r}(\Omega;\mathcal{G})$ be fixed and $(p_k)_k \subset \mathrm{BV}(\Omega;\{0;1\})$ weakly* convergent to $p \in \mathrm{BV}(\Omega;\{0;1\})$. Then, we have

$$\int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{S_{p_k}} \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

Define $\varphi : \Omega \times \mathbb{R}^n \to \mathbb{R}$ as

$$\varphi(x,\mathbf{v}) = \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle^{1/2}.$$

As $r > n$, then we have $W^{1,r}(\Omega) \subset \mathcal{C}(\Omega)$ and then $\mathbf{M}$ is continuous. We deduce that $\varphi : \Omega\times\mathbb{R}^n \to \mathbb{R}$ is continuous as well.

According to Theorem 3.3, we have $Dp_k \ll \mathcal{H}^{n-1}\llcorner S_{p_k}$, $Dp \ll \mathcal{H}^{n-1}\llcorner S_p$ and

$$\frac{\mathrm{d}(Dp_k)}{\mathrm{d}(\mathcal{H}^{n-1}\llcorner S_{p_k})} = \boldsymbol{\nu}_{p_k}\mathbf{1}_{S_{p_k}}, \qquad \frac{\mathrm{d}(Dp)}{\mathrm{d}(\mathcal{H}^{n-1}\llcorner S_p)} = \boldsymbol{\nu}_p\mathbf{1}_{S_p}.$$

Moreover, φ is sublinear with respect to $\mathbf{v}$. According to Proposition 3.3, we can conclude the proof of the *First Step*.

Step 2. Let $(p_k,\mathbf{M}_k)_k \subset \mathcal{Y}$ such as $(p_k)_k$ weakly* converges to $p \in \mathrm{BV}(\Omega;\{0;1\})$ and $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M} \in W^{1,r}(\Omega;\mathcal{G})$. Then, we have

$$\int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{S_{p_k}} \langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

As

$$\left|\langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} - \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2}\right| \leq \frac{|\langle (\mathbf{M}_k - \mathbf{M})\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle|}{\langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} + \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{\frac{1}{2}}}$$

and since Lemma 3.1 gives $\langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} + \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{\frac{1}{2}} \geq 2$, we have

$$\left|\langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} - \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2}\right| \leq \frac{\|\mathbf{M}_k - \mathbf{M}\|_{L^\infty}}{2},$$

which gives that

$$\left|\int_{S_{p_k}} \langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} - \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}\right| \leq \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \frac{\mathcal{H}^{n-1}(S_{p_k})}{2}. \tag{4.2}$$

As $(p_k)_k$ weakly* converges to p in $\mathrm{BV}(\Omega)$, Theorem 3.2 implies that $(\mathcal{H}^{n-1}(S_{p_k}))_k$ is a bounded sequence. Moreover, since $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M}$ in $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R}))$ and since, according to (3.1), the inclusion $W^{1,r}(\Omega;\mathcal{S}_n(\mathbb{R})) \subset \mathcal{C}(\Omega;\mathcal{S}_n(\mathbb{R}))$ is compact, we have that $\|\mathbf{M}_k - \mathbf{M}\|_{L^\infty}$ converges to 0. So, we have the limit

$$\int_{S_{p_k}} \langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} - \int_{S_{p_k}} \langle \mathbf{M}\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \to 0. \tag{4.3}$$

According to *First Step* and (4.3), we can conclude that

$$\int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{S_{p_k}} \langle \mathbf{M}_k\boldsymbol{\nu}_{p_k}, \boldsymbol{\nu}_{p_k}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

□

We can now prove the existence of solutions for problem $(\mathcal{P})$ (4.1).

Theorem 4.1. *The problem* $(\mathcal{P})$ *admits at least one solution.*

Proof. As E is bounded from below by 0, there exists a sequence $(p_k,\mathbf{M}_k)_k \subset \mathcal{Y}$ such that $(E(p_k,\mathbf{M}_k))_k$ converges to the minimum value of E. According to Proposition 4.1, there exists a subsequence, still denoted by $(p_k,\mathbf{M}_k)_k$ which converges to $(p,\mathbf{M}) \in \mathcal{Y}$ for the topology $\mathcal{T}$. According to Proposition 4.2, we have

$$E(p,\mathbf{M}) \leq \liminf_{k\to\infty} E(p_k,\mathbf{M}_k).$$

As $(p_k,\mathbf{M}_k)_k$ is a minimizing sequence for E, we can conclude that $(p,\mathbf{M})$ is a solution of $(\mathcal{P})$. □

5 Approximation process

In this section we give our main result, that is, we introduce an approximate problem and prove the Γ-convergence of this process. This is reminiscent of the Ambrosio-Tortorelli approximation of the Mumford-Shah functional (see [9]).

5.1 Γ-convergence

We want to perform an approximation of the energy E more suitable for numerical applications. This approximation will be a *good one* in the sense of Γ-convergence. In this section, we recall the definition and a useful property (see [10]).

Definition 5.1. *Let $(\mathcal{X}, d)$ be a metrizable space, $(E_k)_k$ a sequence of real-valued functions $E_k : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$, and $E : \mathcal{X} \to \mathbb{R} \cup \{+\infty\}$. The sequence $(E_k)_k$ Γ-converges to E at $x \in \mathcal{X}$ if both of the following conditions hold:*

i) for all sequences $(x_k)_k$ converging to $x \in \mathcal{X}$, one has

$$E(x) \leq \liminf_{k\to\infty} E_k(x_k), \tag{5.1}$$

ii) there exists a sequence $(y_k)_k$ converging to $x \in \mathcal{X}$ such that

$$E(x) \geq \limsup_{k\to\infty} E_k(y_k). \tag{5.2}$$

When (i) and (ii) hold for all $x \in \mathcal{X}$, we say that $(E_k)_k$ Γ-converges to E in $(\mathcal{X}, d)$.

The main interest of Γ-convergence, in our case, is the following result.

Proposition 5.1. *Let $(E_k)_k$ be a sequence of functions which Γ-converges to E in $(\mathcal{X}, d)$. Let $(x_k)_k$ be such that*

$$E_k(x_k) \leq \inf_{x\in\mathcal{X}} E_k(x) + \varepsilon_k,$$

for all k, where $(\varepsilon_k)_k$ converges to 0^+. Assuming that $(x_k)_k$ is relatively compact; then every cluster point x of $(x_k)_k$ is a minimizer of E and

$$\liminf_{k\to\infty} E_k(x_k) = E(x).$$

5.2 The main result

In all the sequel $(\varepsilon_k)_k$ is a sequence of positive numbers converging to 0. Let us introduce the functionals spaces for the approximation process:

- $W^{1,2}(\Omega; [0;1]) = \left\{p \in W^{1,2}(\Omega) \colon\ 0 \leq p(x) \leq 1 \text{ a.e. } x \in \Omega\right\}$,
- $\mathcal{Z} = W^{1,2}(\Omega; [0;1]) \times W^{1,r}(\Omega; \mathcal{G})$.

Let H, F, F_ε and E_ε be the functions defined on $\mathcal{X}$ with values in $[0; +\infty]$ defined by

$$\begin{aligned}
H(p, \mathbf{M}) &= \int_\Omega (p-g)^2 \mathrm{d}x + \|\mathbf{M}\|_{W^{1,r}(\Omega)}, \\
F(p, \mathbf{M}) &= \begin{cases} \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} & \text{if } (p, \mathbf{M}) \in \mathcal{Y}, \\ +\infty & \text{otherwise}, \end{cases} \\
F_\varepsilon(p, \mathbf{M}) &= \begin{cases} \int_\Omega \left(9\varepsilon\langle \mathbf{M}\nabla p, \nabla p\rangle + \frac{p^2(1-p)^2}{\varepsilon}\right) \mathrm{d}x & \text{if } (p, \mathbf{M}) \in \mathcal{Z}, \\ +\infty & \text{otherwise}, \end{cases} \\
E_\varepsilon &= H + F_\varepsilon.
\end{aligned}$$

The following property shows that the domain $\mathcal{Z} \subset \mathcal{X}$ is adapted for the approximation process.

Proposition 5.2. *Let $(p_k, \mathbf{M}_k)_k \subset \mathcal{Z}$ be a sequence converging to $(p, \mathbf{M}) \in \mathcal{X}$ for the topology $\mathcal{T}$ and such that $(E_{\varepsilon_k}(p_k, \mathbf{M}_k))_k$ is a bounded sequence. Then, we have $(p, \mathbf{M}) \in \mathcal{Y}$.*

Proof. It suffices to prove that $p \in BV(\Omega)$. According to Lemma 3.1, we have $|\nabla p_k|^2 \leq \langle \mathbf{M}_k \nabla p_k, \nabla p_k \rangle$, it gives

$$\int_\Omega \left(9\varepsilon_k |\nabla p_k|^2 + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \mathrm{d}x \leq E_{\varepsilon_k}(p_k, \mathbf{M}_k). \tag{5.3}$$

We apply the inequality $2ab \leq a^2 + b^2$ with $a^2 = 9\varepsilon_k |\nabla p_k|^2$ and $b^2 = p_k^2(1-p_k)^2/\varepsilon_k$ to get

$$\int_\Omega |\nabla p_k| p_k (1-p_k) \mathrm{d}x \leq E_{\varepsilon_k}(p_k, \mathbf{M}_k).$$

The left hand side of the inequality is the total variation of $u_k = p_k^2/2 - p_k^3/3$, that is

$$\int_\Omega |\nabla u_k| \mathrm{d}x \leq E_{\varepsilon_k}(p_k, \mathbf{M}_k).$$

Since the right hand side is a bounded, it follows that $(u_k)_k$ is a bounded sequence in $BV(\Omega)$. According to Theorem 3.2, there exists a subsequence which converges weakly* and almost everywhere to $u \in BV(\Omega)$. By assumption, $(p_k)_k$ converges almost everywhere to p, so by the uniqueness of the limit, we have $u = p(1-p)$. Since p takes its values in $\{0; 1\}$, we have $u = p/6$ and $p \in BV(\Omega)$. $\square$

The main result of this work is the following

Theorem 5.1. *The functionals $(E_{\varepsilon_k})_k$ Γ-converge to E in $\mathcal{X}$ for the topology $\mathcal{T}$.*

This results consists in proving the two inequalities (5.1) and (5.2). The first inequality consists in the application of the method introduced in [11], while the second one is specific to this problem.

5.3 The inequality for the lower Γ-limit (5.1)

We now prove inequality (5.1) of the Γ-convergence result in Theorem 5.1. For any $(p, \mathbf{M}) \in \mathcal{X}$, we denote

$$E_-(p, \mathbf{M}) = \inf \left\{ \liminf_{k\to\infty} E_{\varepsilon_k}(p_k, \mathbf{M}_k) \colon \ (p_k, \mathbf{M}_k)_k \subset \mathcal{Z}, \quad (p_k, \mathbf{M}_k) \xrightarrow{\mathcal{T}} (p, \mathbf{M}) \right\}.$$

Let $(p, \mathbf{M}) \in \mathcal{X}$ be such that $E_-(p, \mathbf{M}) < +\infty$. Let us consider a sequence $(p_k, \mathbf{M}_k)_k \subset \mathcal{Z}$ which converges to $(p, \mathbf{M})$ for $\mathcal{T}$. It suffices to prove that $\liminf_{k\to\infty} E_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq E(p, \mathbf{M})$. With no loss of generality, several assumptions may be made.

i) According to Proposition 5.2, $E_-(p, \mathbf{M}) < +\infty$ gives $p \in BV(\Omega)$. So, we may assume $(p, \mathbf{M}) \in \mathcal{Y}$.

ii) The function $q \to E_{\varepsilon_k}(q, \mathbf{M}_k)$ is continuous with respect to the Sobolev norm $W^{1,2}(\Omega)$ and $\mathcal{C}^\infty \cap W^{1,2}(\Omega)$ is dense. So, by diagonal extraction, we may assume that $(p_k)_k \subset \mathcal{C}^\infty(\Omega)$.

As in the proof of Proposition 4.2, we first calculate the limit with $\mathbf{M}$ fixed and then prove a uniform convergence result for $(\mathbf{M}_k)_k$.

Step 1. We will show that $\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}) \geq F(p, \mathbf{M})$.

For any $k \geq 0$, the inequality $a^2 + b^2 \geq 2ab$ gives

$$\int_\Omega \left(9\varepsilon_k \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \mathrm{d}x \geq \int_\Omega 6p_k(1-p_k) \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle^{1/2} \mathrm{d}x.$$

Let $\Phi : \Omega \times [0; 1] \times \mathbb{R}^n \to \mathbb{R}^+$ be the function

$$\Phi(x, s, \mathbf{v}) = 6s(1-s) \langle \mathbf{M}(x) \mathbf{v}, \mathbf{v} \rangle^{1/2}.$$

This function is sublinear in $\mathbf{v}$. We denote $S_t^k = \{x \in \Omega \colon p_k(x) < t\}$ Using the Proposition 3.4, we can write

$$\int_\Omega 6p_k(1-p_k) \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle^{1/2} \mathrm{d}x = \int_{\mathbb{R}} \left(\int_\Omega 6t(1-t) \langle \mathbf{M} D\mathbf{1}_{S_t^k}, D\mathbf{1}_{S_t^k} \rangle^{1/2} \right) \mathrm{d}t.$$

Applying Fatou lemma and noting that $D\mathbf{1}_{S_t}$ vanishes when $t \notin [0;1]$ gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq \int_0^1 6t(1-t) \liminf_{k\to\infty} \left(\int_\Omega \langle \mathbf{M} D\mathbf{1}_{S_t^k}, D\mathbf{1}_{S_t^k} \rangle^{1/2} \right) \mathrm{d}t.$$

As the left hand side of this inequality is finite, for almost every $t \in [0;1]$ we have

$$\liminf_{k\to\infty} \int_\Omega \langle \mathbf{M} D\mathbf{1}_{S_t^k}, D\mathbf{1}_{S_t^k} \rangle^{1/2} < +\infty.$$

Now, Lemma 3.1 gives

$$\int_\Omega |D\mathbf{1}_{S_t^k}| \leq \int_\Omega \langle \mathbf{M} D\mathbf{1}_{S_t^k}, D\mathbf{1}_{S_t^k} \rangle^{1/2},$$

so $\|D\mathbf{1}_{S_t^k}\|_{\mathcal{M}}$ is bounded. This yields that $(\mathbf{1}_{S_t^k})_k$ is relatively weakly* compact in $\mathrm{BV}(\Omega)$. We denote

$$A = \{x \in \Omega \colon p(x) = 1\}$$

and we have:

$$\int_\Omega |p_k - p| \mathrm{d}x = \int_\Omega |p_k - \mathbf{1}_A| \mathrm{d}x \geq \int_{A^c \setminus S_t^k} |p_k - \mathbf{1}_A| \mathrm{d}x + \int_{S_t^k \setminus A^c} |p_k - \mathbf{1}_A| \mathrm{d}x.$$

Reacalling that $S_t^k = \{x \in \Omega \colon p_k(x) < t\}$ gives

$$\begin{array}{rl} |p_k(x) - \mathbf{1}_A(x)| \geq t |\mathbf{1}_{S_t^k}(x) - \mathbf{1}_{A^c}(x)| & \text{for all } x \in A^c \setminus S_t^k, \\ |p_k(x) - \mathbf{1}_A(x)| \geq (1-t) |\mathbf{1}_{S_t^k}(x) - \mathbf{1}_{A^c}(x)| & \text{for all } x \in S_t^k \setminus A^c, \end{array}$$

which further gives

$$\begin{aligned} \int_\Omega |p_k - p| \mathrm{d}x &\geq t \int_{A^c \setminus S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_{A^c}| \mathrm{d}x + (1-t) \int_{S_t^k \setminus A^c} |\mathbf{1}_{S_t^k} - \mathbf{1}_{A^c}| \mathrm{d}x, \\ &\geq \min(t, 1-t) \int_{A^c \Delta S_t^k} |\mathbf{1}_{S_t^k} - \mathbf{1}_{A^c}| \mathrm{d}x, \\ &\geq \min(t, 1-t) \int_\Omega |\mathbf{1}_{S_t^k} - \mathbf{1}_{A^c}| \mathrm{d}x. \end{aligned}$$

So, for any $t \in]0;1[$, the unique possible limit of $(\mathbf{1}_{S_t^k})_k$ is $\mathbf{1}_{A^c}$. As a result, Proposition 3.3 yields

$$\liminf_{k\to\infty} \int_\Omega \left(9\varepsilon_k \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \mathrm{d}x \geq \int_\Omega \langle \mathbf{M} D\mathbf{1}_{A^c}, D\mathbf{1}_{A^c} \rangle^{1/2}.$$

As $D\mathbf{1}_{A^c} = -D\mathbf{1}_A$, we have

$$\liminf_{k\to\infty} \int_\Omega \left(9\varepsilon_k \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \mathrm{d}x \geq \int_\Omega \langle \mathbf{M} D\mathbf{1}_A, D\mathbf{1}_A \rangle^{1/2}.$$

According to Theorem 3.4, we further have $\int_\Omega \langle \mathbf{M} D\mathbf{1}_A, D\mathbf{1}_A \rangle^{1/2} = \int_{S_p} \langle \mathbf{M} \boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$ and then the result of the First Step is proved.

Step 2. We will show that $\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq F(p, \mathbf{M})$. We first prove that

$$\varepsilon_k \left(\int_\Omega \langle \mathbf{M}_k \nabla p_k, \nabla p_k \rangle \mathrm{d}x - \int_\Omega \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle \mathrm{d}x \right) \to 0. \tag{5.4}$$

Since

$$|\langle \mathbf{M}_k \nabla p_k, \nabla p_k \rangle - \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle| \leq \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} |\nabla p_k|^2,$$

we have

$$\varepsilon_k \left| \int_\Omega \langle \mathbf{M}_k \nabla p_k, \nabla p_k \rangle \mathrm{d}x - \int_\Omega \langle \mathbf{M} \nabla p_k, \nabla p_k \rangle \mathrm{d}x \right| \leq \|\mathbf{M}_k - \mathbf{M}\|_{L^\infty} \varepsilon_k \int_\Omega |\nabla p_k|^2 \mathrm{d}x.$$

According to (5.3), the term $\varepsilon_k \int_\Omega |\nabla p_k|^2$ is uniformly bounded with respect to k. Moreover, $(\mathbf{M}_k)_k$ weakly converges to $\mathbf{M}$ and the inclusion $W^{1,r} \subset L^\infty$ is compact. This yields that $(\mathbf{M}_k)_k$ converges to $\mathbf{M}$ in L^∞ and it concludes the the proof of (5.4). We now write $F_{\varepsilon_k}(p_k, \mathbf{M}_k)$ as

$$F_{\varepsilon_k}(p_k, \mathbf{M}_k) = (F_{\varepsilon_k}(p_k, \mathbf{M}_k) - F_{\varepsilon_k}(p_k, \mathbf{M})) + F_{\varepsilon_k}(p_k, \mathbf{M}).$$

According to the Step 1 and (5.4) we can conclude that

$$\liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq F(p, \mathbf{M}).$$

Conclusion. Since $E_{\varepsilon_k} = H + F_{\varepsilon_k}$, we have

$$\liminf_{k\to\infty} E_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq \liminf_{k\to\infty} H(p_k, \mathbf{M}_k) + \liminf_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}_k).$$

According to the Step 2, we have $\liminf F_{\varepsilon_k}(p_k, \mathbf{M}_k) \geq F(p, \mathbf{M})$. Moreover, as H is lower semi-continuous for the topology of $\mathcal{X}$, we have $\liminf H(p_k, \mathbf{M}_k) \geq H(p, \mathbf{M})$. Since $E = F + H$, this finishes the proof of the inequality for the lower Γ-limit.

5.4 The inequality for the upper Γ-limit (5.2)

We now prove inequality (5.2) of the Γ-convergence result in Theorem 5.1.

Proof. We fix $(p, \mathbf{M}) \in \mathcal{X}$. If $p \notin \mathrm{BV}(\Omega)$ then $E(p, \mathbf{M}) = +\infty$. So, we may assume that p belongs to $\mathrm{BV}(\Omega)$ and that it takes its values on $\{0; 1\}$, otherwise the result is ensured. Let $(\varepsilon_k)_k$ be a sequence which converges to 0^+. We construct a sequence of functions $(p_k, \mathbf{M}_k)_k$ converging to $(p, \mathbf{M})$ for the topology $\mathcal{T}$ such that

$$\limsup_{k\to\infty} E_{\varepsilon_k}(p_k, \mathbf{M}_k) \leq E(p, \mathbf{M}).$$

First, we assume that S_p is a smooth surface and that $\mathbf{M}$ is a smooth field of matrices. Then, we remove these assumptions and we use approximating results to give the proof in the general setting.

Step 1. We assume that S_p is a compact surface of class $\mathcal{C}^2$ and $\mathbf{M} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$. In this step, we set $\mathbf{M}_k = \mathbf{M}$ for any k. Moreover, if $(p_k)_k \subset W^{1,2}(\Omega; [0;1])$ converges almost everywhere to p, then it converges in the $L^2(\Omega)$-norm and

$$\left(\int (p_k - g)^2 \mathrm{d}x \right)_k$$

converges to $\int (p - g)^2 \mathrm{d}x$. So, it suffices to construct an appropriate sequence $(p_k)_k$ which converges almost everywhere to p and is such that $\limsup F_{\varepsilon_k}(p_k, \mathbf{M}) \leq F(p, \mathbf{M})$.

For $\eta > 0$, we introduce the following set V_η.

$$V_\eta = \{x \in [p = 1] \colon 0 < \mathrm{dist}(x, S_p) < \eta\}.$$

Outside V_η, we define the function p_k as:

$$p_k(x) = \begin{cases} 0 & \text{for all } x \in [p = 0], \\ 1 & \text{for all } x \in [p = 1] \setminus V_\eta. \end{cases}$$

The construction of p_k inside V_η will be made precise. Since we assume that S_p is a compact and $\mathcal{C}^2$ surface, there exists $\eta_0 > 0$ and a $\mathcal{C}^1$-diffeomorphism $\phi : V_{\eta_0} \to S_p \times]0; \eta_0[$ (see [8]), characterized by

$$\phi(\xi + t\boldsymbol{\nu}_p(\xi)) = (\xi, t)$$

for all $(\xi, t) \in S_p \times]0; \eta_0[$.

We denote by Σ_ξ the slice

$$\Sigma_\xi = \{\xi + t\boldsymbol{\nu}_p(\xi) \colon t \in [0; \eta_0]\}.$$

We shall construct p_k slice by slice (see figure 5.1). Indeed, $\phi : V_{\eta_0} \to S_p \times]0; \eta_0[$ is a diffeomorphism, so it provides a complete construction of p_k. We denote by $\chi_{k,\xi} : [0; \eta_0] \to \mathbb{R}$ the restriction of p_k to Σ_ξ. We introduce K defined on $S_p \times [0; \eta_0]$ by

$$K(\xi, t) = \langle \mathbf{M}(\xi + t\boldsymbol{\nu}_p(\xi))\, \boldsymbol{\nu}_p(\xi), \boldsymbol{\nu}_p(\xi) \rangle^{1/2}$$

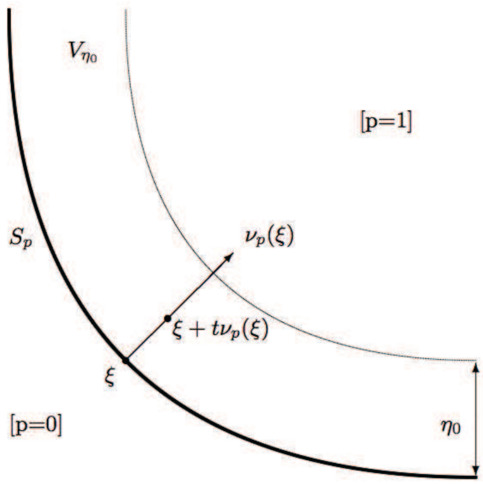

Figure 5.1: Slicing parametrization of V_{η_0}

for all $(\xi, t) \in S_p \times [0; \eta_0]$. We define $\chi_{k,\xi}$ as the solution of the following differential equation

$$3\sqrt{\varepsilon_k} K(\xi,t)\chi'_{k,\xi}(t) = \left(\frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2 (1-\chi_{k,\xi}(t))^2}{\varepsilon_k} \right)^{1/2} \quad \text{for all } t \geq 0,$$

with initial condition $\chi_{k,\xi}(0) = 0$.

Remark 5.1. *This definition may appear as "a rabbit pulled out of a hat". Actually, for $\xi \in S_p$ fixed, we have considered a critical point of the minimizing problem restricted to the slice Σ_ξ. Solving the corresponding Euler equation in one dimension has led us to consider the differential equation*

$$3\sqrt{\varepsilon_k} K(\xi,t)\chi'_{k,\xi}(t) = \left(c_k + \frac{(\chi_{k,\xi}(t))^2 (1-\chi_{k,\xi}(t))^2}{\varepsilon_k} \right)^{1/2},$$

where c_k was a constant to be determined. The following analysis has forced us to fix $c_k = 1/\varepsilon_k |\ln(\varepsilon_k)|$.

For $t \geq 0$, we have

$$\chi'_{k,\xi}(t) \geq \frac{1}{3K(\xi,t)\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}}.$$

According to Lemma 3.1, we have $K(\xi,t) \leq \sqrt{2}$. So, there exists a unique $\eta_{k,\xi} > 0$ such that $\chi_{k,\xi}(\eta_{k,\xi}) = 1$ and such that it satisfies

$$\sup_{\xi \in S_p} \eta_{k,\xi} \leq 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}. \tag{5.5}$$

As $\varepsilon_k \sqrt{|\ln(\varepsilon_k)|}$ converges to 0, then we can assume that $\eta_{k,\xi} < \eta_0$ for any k and $\xi \in S_p$. Thus, we modify the definition of $\chi_{k,\xi}$ as the solution of the following equation

$$\left\{ \begin{array}{rcll} 3\sqrt{\varepsilon_k} K(\xi,t)\chi'_{k,\xi}(t) &=& \left(\frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2 (1-\chi_{k,\xi}(t))^2}{\varepsilon_k} \right)^{1/2} & \text{for all } t \in]0; \eta_{k,\xi}[, \\ \chi_{k,\xi}(t) &=& 1 & \text{for all } t \in [\eta_{k,\xi}; \eta_0[, \\ \chi_{k,\xi}(0) &=& 0. & \end{array} \right. \tag{5.6}$$

We denote $\eta_k = \sup\{\eta_{k,\xi} : \xi \in S_p\}$ and we define p_k as

$$\left\{ \begin{array}{rcll} p_k(x) &=& 0 & \text{for all } x \in [p=0], \\ p_k(\xi + t\boldsymbol{\nu}_p(\xi)) &=& \chi_{k,\xi}(t) & \text{for all } (\xi,t) \in S_p \times]0; \eta_0[, \\ p_k(x) &=& 1 & \text{for all } x \in [p=1] \setminus V_{\eta_0}. \end{array} \right. \tag{5.7}$$

According to (5.5), we have $\eta_k \to 0$ which implies $p_k \to p$ almost everywhere. With the definitions introduced in (5.6) and (5.7), we have to prove that $\limsup F_{\varepsilon_k}(p_k, \mathbf{M}) \leq F(p, \mathbf{M})$. In the sequel we take $n = 3$ but the arguments are the same for $n = 2$. As S_p is a surface of class $\mathcal{C}^2$, there exist two functions $\mathbf{t}_1$ and $\mathbf{t}_2$ defined in S_p taking their values in the unit sphere $\mathbb{S}^{n-1}$ and of class $\mathcal{C}^1$ such that, for any $\xi \in S_p$, the vector triplet $(\mathbf{t}_1(\xi), \mathbf{t}_2(\xi), \boldsymbol{\nu}_p(\xi))$ is an orthonormal basis of $\mathbb{R}^3$.

We need to prove the following lemma.

Lemma 5.1. *For any $\varepsilon > 0$ and $\mathbf{v} \in \mathbb{R}^n$, we have*

$$\langle \mathbf{M}\mathbf{v}, \mathbf{v}\rangle \leq \langle \mathbf{v}, \boldsymbol{\nu}_p\rangle^2 \left(\langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle + 2\varepsilon\right) + \langle \mathbf{v}, \mathbf{t}_1\rangle^2 \left(3 + \frac{1}{\varepsilon}\right) + \langle \mathbf{v}, \mathbf{t}_2\rangle^2 \left(3 + \frac{1}{\varepsilon}\right),$$

where $\boldsymbol{\nu}_p, \mathbf{t}_1, \mathbf{t}_2$ depend on $\xi \in S_p$ and $\mathbf{M}$ depends on $x \in \Omega$.

Proof. As $\mathbf{M}$ takes its values in $\mathcal{G}$, there exists $\mathbf{c} : \Omega \to \mathbb{S}^{n-1}$ such that $\mathbf{M} = \mathrm{Id}_n + \mathbf{c}\mathbf{c}^t$. We have the decomposition

$$\left\{ \begin{array}{rcl} |\mathbf{v}|^2 &=& \langle \mathbf{v}, \boldsymbol{\nu}_p\rangle^2 + \langle \mathbf{v}, \mathbf{t}_1\rangle^2 + \langle \mathbf{v}, \mathbf{t}_2\rangle^2, \\ \langle \mathbf{c}, \mathbf{v}\rangle &=& \langle \mathbf{c}, \boldsymbol{\nu}_p\rangle\langle \mathbf{v}, \boldsymbol{\nu}_p\rangle + \langle \mathbf{c}, \mathbf{t}_1\rangle\langle \mathbf{v}, \mathbf{t}_1\rangle + \langle \mathbf{c}, \mathbf{t}_2\rangle\langle \mathbf{v}, \mathbf{t}_2\rangle. \end{array}\right.$$

We denote $a = \langle \mathbf{c}, \boldsymbol{\nu}_p\rangle\langle \mathbf{v}, \boldsymbol{\nu}_p\rangle$, $b = \langle \mathbf{c}, \mathbf{t}_1\rangle\langle \mathbf{v}, \mathbf{t}_1\rangle$ and $c = \langle \mathbf{c}, \mathbf{t}_2\rangle\langle \mathbf{v}, \mathbf{t}_2\rangle$. Moreover, we have

$$\begin{array}{rcl} (a+b+c)^2 &=& a^2 + b^2 + c^2 + 2ab + 2ac + 2bc, \\ &\leq& a^2 + b^2 + c^2 + \left(\varepsilon a^2 + \dfrac{b^2}{\varepsilon}\right) + \left(\varepsilon a^2 + \dfrac{c^2}{\varepsilon}\right) + \left(b^2 + c^2\right), \\ &\leq& (1 + 2\varepsilon)a^2 + \left(2 + \dfrac{1}{\varepsilon}\right) b^2 + \left(2 + \dfrac{1}{\varepsilon}\right) c^2. \end{array}$$

We may introduce $\langle \mathbf{c}, \mathbf{t}_1\rangle^2 \leq 1$, $\langle \mathbf{c}, \mathbf{t}_2\rangle^2 \leq 1$ in the previous inequality, which gives the result of the lemma. □

If we apply Lemma 5.1 in the definition of F_{ε_k}, we get

$$F_{\varepsilon_k}(p_k, \mathbf{M}) \leq (\star)_k^{\boldsymbol{\nu}_p} + (\star)_k^{\mathbf{t}_1} + (\star)_k^{\mathbf{t}_2}, \tag{5.8}$$

where

$$\left\{ \begin{array}{rcl} (\star)_k^{\boldsymbol{\nu}_p} &=& \int_{V_{\eta_k}} 9\varepsilon_k \langle \nabla p_k, \boldsymbol{\nu}_p\rangle^2 \left(\langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle + 2\varepsilon_k\right) + \frac{p_k^2(1-p_k)^2}{\varepsilon_k}, \\ (\star)_k^{\mathbf{t}_1} &=& \int_{V_{\eta_k}} 9\varepsilon_k \langle \nabla p_k, \mathbf{t}_1\rangle^2 \left(3 + \frac{1}{\varepsilon_k}\right), \\ (\star)_k^{\mathbf{t}_2} &=& \int_{V_{\eta_k}} 9\varepsilon_k \langle \nabla p_k, \mathbf{t}_2\rangle^2 \left(3 + \frac{1}{\varepsilon_k}\right). \end{array}\right. \tag{5.9}$$

To conclude the *First Step* it is sufficient to prove the following assertions

$$\limsup_{k\to\infty} (\star)_k^{\boldsymbol{\nu}_p} \leq \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}, \quad \lim_{k\to\infty} (\star)_k^{\mathbf{t}_1} = 0, \quad \lim_{k\to\infty} (\star)_k^{\mathbf{t}_2} = 0.$$

Remark 5.2. *Roughly speaking, we will show that the energy is totally supported in the limit by the normal component (in the direction of ν_p). The lateral components (in the direction of $\mathbf{t}_1$ or $\mathbf{t}_2$) are null.*

Claim 1. *We have the following inequality*

$$\limsup_{k\to\infty} (\star)_k^{\boldsymbol{\nu}_p} \leq \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

We have

- $\langle \nabla p_k, \boldsymbol{\nu}_p\rangle^2 \leq |\nabla p_k|^2$,
- $|\nabla p_k|^2 \leq \langle \mathbf{M}\nabla p_k, \nabla p_k\rangle$, according to Lemma 3.1,
- $\int_\Omega 9\varepsilon_k \langle \mathbf{M}\nabla p_k, \nabla p_k\rangle \mathrm{d}x \leq F_{\varepsilon_k}(p_k, \mathbf{M})$,
- $F_{\varepsilon_k}(p_k, \mathbf{M})$ is bounded.

Then, we can conclude that

$$\int_{V_{\eta_k}} 9\varepsilon_k \langle \nabla p_k, \boldsymbol{\nu}_p \rangle^2 \mathrm{d}x$$

is bounded. In particular, $\varepsilon_k \int_{V_{\eta_k}} 9\varepsilon_k \langle \nabla p_k, \boldsymbol{\nu}_p \rangle^2 \mathrm{d}x$ converges to 0. Then, for Claim 1, it suffices to prove

$$\limsup_{k\to\infty} \int_{V_{\eta_k}} \left(9\varepsilon_k \langle \nabla p_k, \boldsymbol{\nu}_p \rangle^2 \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \mathrm{d}x \leq \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

Since

$$\frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi + t\boldsymbol{\nu}_p(\xi)) = \lim_{s\to 0} \frac{p_k(\xi + (s+t)\boldsymbol{\nu}_p(\xi)) - p_k(\xi + t\boldsymbol{\nu}_p(\xi))}{s} = \lim_{s\to 0} \frac{\chi_{k,\xi}(s+t) - \chi_{k,\xi}(t)}{s},$$

it follows that, for any $(\xi, t) \in S_p \times]0; \eta_k[$, we have

$$\frac{\partial p_k}{\partial \boldsymbol{\nu}_p(\xi)}(\xi + t\boldsymbol{\nu}_p(\xi)) = \chi'_{k,\xi}(t).$$

This yields $\langle \nabla p_k, \boldsymbol{\nu}_p \rangle = \chi'_{k,\xi}$. According to the assumptions of regularity of S_p, as in [8], we may introduce the change of variables

$$\int_{S_p} \int_0^{\eta_0} \frac{\mathrm{d}t \ \mathrm{d}\mathcal{H}^2(\xi)}{(1-\kappa_1(\xi)t)(1-\kappa_2(\xi)t)} = \int_{V_{\eta_0}} \mathrm{d}x,$$

where $\kappa_1(\xi), \kappa_2(\xi)$ are the principal curvatures of S_p at ξ. Since S_p is a $\mathcal{C}^2$ surface, it follows that κ_1 and κ_2 are continuous on S_p. We denote

$$\Pi(\xi, t) = \frac{1}{(1-\kappa_1(\xi)t)(1-\kappa_2(\xi)t)}.$$

This yields

$$\begin{aligned}
(\star)_k^{\boldsymbol{\nu}_p} &= \int_{S_p} \int_0^{\eta_k} \left(9\varepsilon_k \langle \nabla p_k, \boldsymbol{\nu}_p \rangle^2 \langle \mathbf{M}\boldsymbol{\nu}_p, \boldsymbol{\nu}_p \rangle + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \Pi \, \mathrm{d}t \ \mathrm{d}\mathcal{H}^2(\xi), \\
&= \int_{S_p} \int_0^{\eta_k} \left(9\varepsilon_k (\chi'_{k,\xi})^2 K^2 + \frac{p_k^2(1-p_k)^2}{\varepsilon_k} \right) \Pi \, \mathrm{d}t \ \mathrm{d}\mathcal{H}^2(\xi).
\end{aligned}$$

In these integrals we remove the dependent variables for the sake of simplicity, that is,

$$x = \xi + t\boldsymbol{\nu}_p(\xi), \quad \boldsymbol{\nu}_p = \boldsymbol{\nu}_p(\xi), \quad p_k = p_k(\xi + t\boldsymbol{\nu}_p(\xi)), \quad \Pi = \Pi(\xi, t), \chi_{k,\xi} = \chi_{k,\xi}(t), \quad K = K(\xi, t)$$

and we set

$$a = 3\sqrt{\varepsilon_k} K \chi'_{k,\xi}, \qquad b = \frac{\chi_{k,\xi}(1-\chi_{k,\xi})}{\sqrt{\varepsilon_k}}.$$

With the construction of $\chi_{k,\xi}$ in (5.6) we get $a^2 = \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + b^2$ so that $0 \leq b \leq a$ on $[0; \eta_k]$ and

$$a^2 + b^2 \leq 2ab + \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|}.$$

This yields

$$(\star)_k^{\boldsymbol{\nu}_p} \leq \underbrace{\int_{S_p} \int_0^{\eta_k} 6K \chi'_{k,\xi} \chi_{k,\xi}(1-\chi_{k,\xi}) \Pi \, \mathrm{d}t \ \mathrm{d}\mathcal{H}^2(\xi)}_{\left((\star)_k^{\boldsymbol{\nu}_p}\right)_A} + \underbrace{\int_{S_p} \int_0^{\eta_k} \frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} \Pi \, \mathrm{d}t \ \mathrm{d}\mathcal{H}^2(\xi)}_{\left((\star)_k^{\boldsymbol{\nu}_p}\right)_B}.$$

The functions K and Π are uniformly bounded with respect to k in $S_p \times]0; \eta_0[$. We denote by m their upper bound. We have the following inequality

$$\left((\star)_k^{\boldsymbol{\nu}_p}\right)_B \leq m\mathcal{H}^2(S_p) \frac{\eta_k}{\varepsilon_k |\ln(\varepsilon_k)|}.$$

According to (5.5), we have $\eta_k \leq 3\sqrt{2}\varepsilon_k\sqrt{|\ln(\varepsilon_k)|}$. This yields

$$\left((\star)_k^{\boldsymbol{\nu}_p}\right)_B \quad \leq \frac{3\sqrt{2}m\mathcal{H}^2(S_p)}{\sqrt{|\ln(\varepsilon_k)|}}.$$

and then $\lim \left((\star)_k^{\boldsymbol{\nu}_p}\right)_B = 0$.

Denoting $L(\xi,t) = K(\xi,t)\Pi(\xi,t)$, we have

$$\begin{aligned}\left((\star)_k^{\boldsymbol{\nu}_p}\right)_A \quad &= \underbrace{\int_{S_p}\int_0^{\eta_k} 6(L(\xi,t)-L(\xi,0))\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1}} \\ &\quad + \underbrace{\int_{S_p}\int_0^{\eta_k} 6L(\xi,0)\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi)}_{\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,2}}.\end{aligned}$$

As $\chi'_{k,\xi}\chi_{k,\xi}(1-\chi_{k,\xi})$ is nonnegative, we have

$$\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1} \leq \sup|L(\xi,t)-L(\xi,0)|\int_{S_p}\int_0^{\eta_k} 6\chi'_{k,\xi}(t)\chi_{k,\xi}(t)(1-\chi_{k,\xi}(t))\mathrm{d}t\ \mathrm{d}\mathcal{H}^2(\xi),$$

where the sup is taken over all $(\xi,t) \in S_p\times]0;\eta_k[$. Since $\chi_{k,\xi} \in W^{1,2}(]0;\eta_k[)$, we may use the change of variable $s = \chi_{k,\xi}(t)$ to obtain

$$\begin{aligned}\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1} \quad &\leq \quad \sup|L(\xi,t)-L(\xi,0)|\int_{S_p}\int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} 6s(1-s)\mathrm{d}s\ \mathrm{d}\mathcal{H}^2(\xi), \\ &\leq \quad \sup|L(\xi,t)-L(\xi,0)|\,\mathcal{H}^2(S_p).\end{aligned}$$

The surface S_p is compact and smooth and the function L is continuous. As a result, the family $(L(\cdot,t))_{t>0}$ uniformly converges to $L(\cdot,0)$ when $t \to 0^+$. We can deduce that $\lim \left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1} = 0$. Using the same change of variable $s = \chi_{k,\xi}(t)$ in $\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,2}$ gives

$$\begin{aligned}\left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,2} \quad &= \quad \int_{S_p} L(\xi,0)\int_{\chi_{k,\xi}(0)}^{\chi_{k,\xi}(\eta_k)} 6s(1-s)\mathrm{d}s\ \mathrm{d}\mathcal{H}^2(\xi), \\ &= \quad \int_{S_p}\langle\mathbf{M}(\xi)\boldsymbol{\nu}_p(\xi),\boldsymbol{\nu}_p(\xi)\rangle^{1/2}\mathrm{d}\mathcal{H}^2(\xi).\end{aligned}$$

To summarize, we have the decomposition

$$(\star)_k^{\boldsymbol{\nu}_p} = \left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1} + \left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,2} + \left((\star)_k^{\boldsymbol{\nu}_p}\right)_B$$

and these terms satisfy

$$\lim \left((\star)_k^{\boldsymbol{\nu}_p}\right)_B = 0, \quad \lim \left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,1} = 0, \quad \left((\star)_k^{\boldsymbol{\nu}_p}\right)_{A,2} = \int_{S_p}\langle\mathbf{M}\boldsymbol{\nu}_p,\boldsymbol{\nu}_p\rangle^{1/2}\mathrm{d}\mathcal{H}^2.$$

This concludes the proof of Claim 1.

Claim 2. *We have the limits*

$$\lim_{k\to\infty}(\star)_k^{\mathbf{t}_1} = 0, \quad \lim_{k\to\infty}(\star)_k^{\mathbf{t}_2} = 0,$$

where $(\star)_k^{\mathbf{t}_1}$ *and* $(\star)_k^{\mathbf{t}_2}$ *are introduced in (5.9).*

We will prove the result for $(\star)_k^{\mathbf{t}_1}$, since the method for $(\star)_k^{\mathbf{t}_2}$ is the same. As S_p is a $\mathcal{C}^2$ surface, the intersection of the affine plane $P_1 = \xi + \mathrm{Vect}(\mathbf{t}_1(\xi),\boldsymbol{\nu}_p(\xi))$ and S_p at the neighborhood of $\xi \in S_p$ is a $\mathcal{C}^2$-planar curve. Let I be a neighborhood of 0 in $\mathbb{R}$ and $\gamma : I \to S_p$ be a local curvilinear parametrization of this curve such that

$$\begin{cases}\gamma(0) = \xi, \\ \gamma'(0) = \mathbf{t}_1(\xi), \\ |\gamma'(t)| = 1, \text{ for all } t \in I.\end{cases}$$

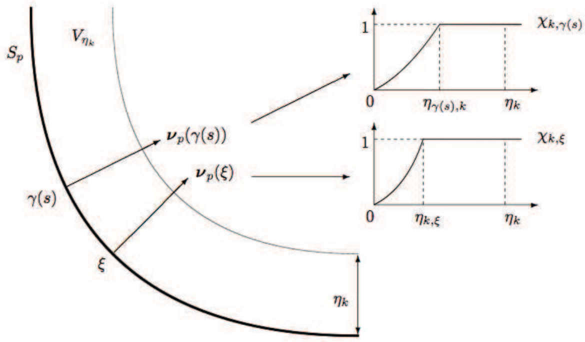

Figure 5.2: Construction of the solution on each slice

As $\boldsymbol{\nu}_p \circ \gamma(s)$ is orthogonal to $\gamma'(s)$ for all $s \in I$ and γ is a planar curve, there exists $\tilde{\kappa}_1 : I \to \mathbb{R}$ such that

$$\frac{\mathrm{d}(\boldsymbol{\nu}_p \circ \gamma)}{\mathrm{d}s}(s) = -\tilde{\kappa}_1(\gamma(s))\mathbf{t}_1(\gamma(s)).$$

As γ is a curve of S_p, and $\tilde{\kappa}_1$ is the sectional curvature of S_p in the direction of $\mathbf{t}_1(\gamma(s))$, we have $|\tilde{\kappa}_1| \leq \max(|\kappa_1|, |\kappa_2|)$.

We evaluate

$$\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t) \quad = \quad p_k(\gamma(s) + t\boldsymbol{\nu}_p(\gamma(s))) - p_k(\xi + t\boldsymbol{\nu}_p(\xi))).$$

So, at $s = 0$, we have the asymptotic expansion

$$\gamma(s) + t\boldsymbol{\nu}_p(\gamma(s)) = \xi + t\boldsymbol{\nu}_p(\xi) + s(1 - \tilde{\kappa}_1(\xi)t)\mathbf{t}_1(\xi) + o(s)$$

and we get

$$\lim_{s\to 0} \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s(1 - \tilde{\kappa}_1(\xi)t)} \quad = \quad \langle \nabla p_k(\xi + t\boldsymbol{\nu}_p(\xi)), \mathbf{t}_1(\xi)\rangle. \tag{5.10}$$

In order to calculate the left hand side of (5.10), we denote by $(E)_\xi$ the equation satisfied by $\chi_{k,\xi}$ (see figure 5.2) and we recall that

$$(E)_\xi : \left\{ \begin{array}{rcll} 3\sqrt{\varepsilon_k}K(\xi,t)\chi'_{k,\xi}(t) &=& \left(\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{(\chi_{k,\xi}(t))^2(1-\chi_{k,\xi}(t))^2}{\varepsilon_k}\right)^{1/2} & \text{for all } t \in]0;\eta_{k,\xi}[, \\ \chi_{k,\xi}(t) &=& 1 & \text{for all } t \geq \eta_{k,\xi}, \\ \chi_{k,\xi}(0) &=& 0, & \end{array} \right.$$

We denote

$$f_k(x) = \left(\frac{1}{\varepsilon_k|\ln(\varepsilon_k)|} + \frac{x^2(1-x)^2}{\varepsilon_k}\right)^{1/2}, \quad Y_{k,s}(t) = \chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t), \quad \overline{\eta}_{k,s} = \min(\eta_{\gamma(s),k}, \eta_{\xi,k}).$$

We calculate

$$\frac{K(\xi,\cdot)}{K(\gamma(s),\cdot)}(E)_{\gamma(s)} - (E)_\xi,$$

which gives

$$\left\{ \begin{array}{rcl} Y_{k,s}(0) &=& 0, \\ 3\sqrt{\varepsilon_k}K(\xi,t)Y'_{k,s}(t) &=& \frac{K(\xi,t)}{K(\gamma(s),t)}f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t)), \ \text{for all } t \in]0;\overline{\eta}_{k,s}[. \end{array} \right.$$

Lemma 3.1 further gives that

$$1 \leq K(\xi,t) \leq \sqrt{2}. \tag{5.11}$$

for all $(\xi,t) \in S_p \times]0;\overline{\eta}_{k,s}[$. Since S_p is a $\mathcal{C}^2$-manifold and $\mathbf{M} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega;\mathcal{G})$, it follows that K is a function of class $\mathcal{C}^1$ and there exists a constant $\tau > 0$ such that

$$|K(\xi,t) - K(\xi',t)| \leq \tau|\xi - \xi'| \tag{5.12}$$

for all $(\xi, \xi', t) \in S_p^2 \times]0; \eta_k[$. Moreover, the study of f_k gives

$$f_k(x) \leq \left(\frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k} \right)^{1/2}, \qquad |f_k'(x)| \leq \frac{1}{\sqrt{\varepsilon_k}} \tag{5.13}$$

for all $x \in [0; 1]$. Using the equation

$$3\sqrt{\varepsilon_k} K(\xi, t) Y_{k,s}'(t) = \frac{K(\xi, t)}{K(\gamma(s), t)} (f_k(\chi_{k,\gamma(s)}(t)) - f_k(\chi_{k,\xi}(t))) + f_k(\chi_{k,\xi}(t)) \left(\frac{K(\xi, t) - K(\gamma(s), t)}{K(\gamma(s), t)} \right)$$

and (5.11), (5.12), (5.13), we get

$$3\sqrt{\varepsilon_k} Y_{k,s}'(t) \leq \frac{\sqrt{2}}{\sqrt{\varepsilon_k}} Y_{k,s}(t) + \tau s \sqrt{\frac{1}{\varepsilon_k |\ln(\varepsilon_k)|} + \frac{1}{16\varepsilon_k}}.$$

Thus, $Y_{k,s}$ is a solution of the differential inequality

$$\begin{cases} Y_{k,s}(0) &= 0, \\ Y_{k,s}'(t) &\leq \frac{\sqrt{2}}{3\varepsilon_k} Y_{k,s}(t) + \frac{\tau s}{3\varepsilon_k} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \text{ for all } t \in]0; \overline{\eta}_{k,s}[. \end{cases} \tag{5.14}$$

So, we have

$$Y_{k,s}(t) \leq \frac{\tau s}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left(\exp\left(\frac{\sqrt{2} t}{3\varepsilon_k} \right) - 1 \right). \tag{5.15}$$

The definition of $Y_{k,s}$ gives

$$\frac{Y_{k,s}(t) - Y_{k,0}(t)}{s} = \frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s}$$

and inequality (5.15) implies that for any $t \in]0; \overline{\eta}_{k,s}[$ we have

$$\frac{\chi_{k,\gamma(s)}(t) - \chi_{k,\xi}(t)}{s} \leq \frac{\tau}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left(\exp\left(\frac{\sqrt{2} t}{3\varepsilon_k} \right) - 1 \right). \tag{5.16}$$

According to the continuous dependance of the solution of the equation (5.14) with respect to the parameter s, then $\overline{\eta}_{k,s}$ converges to $\eta_{k,\xi}$ when s converges to 0. So, the inequality (5.16) remains true in the neighborhood of any point $t \in]0; \eta_{k,\xi}[$. With k, ξ and $t \in]0; \eta_{k,\xi}[$ fixed, we calculate the limit when s converges to 0, and we apply equality (5.10) to get

$$(1 - \tilde{\kappa}_1(\xi) t) \langle \nabla p_k(\xi + t \boldsymbol{\nu}_p(\xi)), \mathbf{t}_1(\xi) \rangle \leq \frac{\tau}{\sqrt{2}} \sqrt{\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16}} \left(\exp\left(\frac{\sqrt{2} t}{3\varepsilon_k} \right) - 1 \right).$$

As $\eta_k \to 0$ and $\tilde{\kappa}_1$ is continuous, there exists $r > 0$ such that

$$r < (1 - \tilde{\kappa}_1(\xi) t)$$

for all $(\xi, t) \in S_p \times]0; \eta_k[$. This gives

$$\langle \nabla p_k(\xi + t \boldsymbol{\nu}_p(\xi)), \mathbf{t}_1(\xi) \rangle^2 \leq \frac{\tau^2}{2r^2} \left(\frac{1}{|\ln(\varepsilon_k)|} + \frac{1}{16} \right) \left(\exp\left(\frac{\sqrt{2} t}{3\varepsilon_k} \right) - 1 \right)^2. \tag{5.17}$$

As $1/|\ln(\varepsilon_k)| \to 0$, there exists $m > 0$ such that (5.17) becomes

$$\langle \nabla p_k(\xi + t \boldsymbol{\nu}_p(\xi)), \mathbf{t}_1(\xi) \rangle^2 \leq m \exp\left(\frac{2\sqrt{2} t}{3\varepsilon_k} \right) + m.$$

As $2\sqrt{2}/3 \leq 1$, we have

$$\langle \nabla p_k(\xi + t \boldsymbol{\nu}_p(\xi)), \mathbf{t}_1(\xi) \rangle^2 \leq m \exp\left(\frac{t}{\varepsilon_k} \right) + m. \tag{5.18}$$

Introducing (5.18) in the definition of $(\star)_k^{\mathbf{t}_1}$ in (5.9) gives

$$(\star)_k^{\mathbf{t}_1} \leq \int_{S_p} \int_0^{\eta_k} 9\varepsilon_k \left(m \exp\left(\frac{t}{\varepsilon_k}\right) + m \right) \left(3 + \frac{1}{\varepsilon_k} \right) \Pi(\xi, t) \mathrm{d}t \; \mathrm{d}\mathcal{H}^2(\xi).$$

Since $\eta_k \to 0$, the function Π is bounded and there exists a positive constant, still denoted by m, such that

$$(\star)_k^{\mathbf{t}_1} \leq \int_{S_p} \int_0^{\eta_k} \varepsilon_k \left(m \exp\left(\frac{t}{\varepsilon_k}\right) + m \right) m \left(3 + \frac{1}{\varepsilon_k} \right) \mathrm{d}t \; \mathrm{d}\mathcal{H}^2(\xi).$$

Thus, we have

$$(\star)_k^{\mathbf{t}_1} \leq \left[\varepsilon_k^2 \left(\exp\left(\frac{\eta_k}{\varepsilon_k}\right) - 1 \right) + \varepsilon_k \eta_k \right] \left(3 + \frac{1}{\varepsilon_k} \right) m^2 \mathcal{H}^2(S_p).$$

The upper bound on η_k in (5.5) implies that

$$\begin{aligned}
(\star)_k^{\mathbf{t}_1} &\leq \left(\varepsilon_k^2 \left(\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}\right) - 1 \right) + 3\sqrt{2}\varepsilon_k^2 \sqrt{|\ln(\varepsilon_k)|} \right) \left(3 + \frac{1}{\varepsilon_k} \right) m^2 \mathcal{H}^2(S_p), \\
(\star)_k^{\mathbf{t}_1} &\leq \left(\varepsilon_k \exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|}\right) - \varepsilon_k + 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \right) (1 + 3\varepsilon_k) m^2 \mathcal{H}^2(S_p), \\
(\star)_k^{\mathbf{t}_1} &\leq \left(\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} + \ln(\varepsilon_k)\right) - \varepsilon_k + 3\sqrt{2}\varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \right) m^2 \mathcal{H}^2(S_p).
\end{aligned}$$

As $\varepsilon_k \to 0^+$, we have

$$\exp\left(3\sqrt{2}\sqrt{|\ln(\varepsilon_k)|} + \ln(\varepsilon_k)\right) \to 0^+, \qquad \varepsilon_k \sqrt{|\ln(\varepsilon_k)|} \to 0^+.$$

We can conclude that $(\star)_k^{\mathbf{t}_1} \to 0$.

Step 2. Assume that $p \in \mathrm{BV}(\Omega)$, p takes its values in $\{0; 1\}$ and $\mathbf{M} \in \mathcal{C}^\infty \cap W^{1,r}(\Omega; \mathcal{G})$. In this step, we still set $\mathbf{M}_k = \mathbf{M}$ for any k. For the same reason as in the previous step, it suffices to construct an appropriate sequence $(p_k)_k$ which converges almost everywhere to p and is such that $\limsup F_{\varepsilon_k}(p_k, \mathbf{M}) \leq F(p, \mathbf{M})$.

We denote $A = p^{-1}(\{1\})$. Let us first assume that A and $\Omega \setminus A$ have nonempty interior. We can apply Lemma 3.2, so there exists a sequence $(A_l)_l$ of open bounded subsets of $\mathbb{R}^n$ with smooth boundaries such that

i) $\lim_{l\to\infty} \mathcal{L}^n((A_l \cap \Omega) \triangle A) = 0$ and $\lim_{l\to\infty} \mathcal{H}^{n-1}(\partial A_l) = \mathcal{H}^{n-1}(\partial A)$;

ii) $\mathcal{L}^n(A_l \cap \Omega) = \mathcal{L}^n(A)$ for l large enough;

iii) $\mathcal{H}^{n-1}(\partial A_l \cap \partial \Omega) = 0$ for l large enough;

iv) we have

$$F(p_l, \mathbf{M}) \leq F(p, \mathbf{M}) + \frac{1}{l}, \tag{5.19}$$

where $\mathcal{L}^n$ is the Lebesgue measure over Ω and $p_l = \mathbf{1}_{A_l \cap \Omega}$. For (5.19) we use the fact that $\|D\mathbf{1}_{A_l}\|_{\mathcal{M}} \to \|D\mathbf{1}_A\|_{\mathcal{M}}$ and Proposition 3.3 (ii). With (i), (ii) and (iii), we can say that $(p_l)_l$ is a bounded sequence of $\mathrm{BV}(\Omega)$ which converges to p in $L^1(\Omega)$. According to Proposition 4.1, there exists a subsequence, still denoted by $(p_l)_l$ which weakly* converges to p in $\mathrm{BV}(\Omega)$. One can apply the result of the first step with $p = p_l$. So, there exists a sequence $(p_{l,k})_k$ which weakly* converges to p_l in $\mathrm{BV}(\Omega)$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_{k,l}, \mathbf{M}) \leq F(p_l, \mathbf{M}). \tag{5.20}$$

With (5.19), (5.20) and a diagonal extraction there exists a sequence $(p_k)_k$ which weakly* converges to p such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k, \mathbf{M}) \leq F(p, \mathbf{M}).$$

Let us remove the restriction that both A or $\Omega \setminus A$ have non empty interior. First, we notice that if $\mathcal{L}^n(A) = 0$ or $\mathcal{L}^n(A) = \Omega$ the result is obvious by taking $A_l = \emptyset$ or $A_l = \Omega$ for all l. So, we may assume that $0 < \mathcal{L}^n(A) < |\Omega|$. There exist two points x_1, x_2 such that

- $x_1 \in A$ and for all $r > 0$ there holds $\mathcal{L}^n(A \cap B(x_1, r)) > 0$,
- $x_2 \in \Omega \setminus A$ and for all $r > 0$ there holds $\mathcal{L}^n((\Omega \setminus A) \cap B(x_1, r)) > 0$.

Consider the set $A_{\theta_1,\theta_2} = (A \cup B(x_2,\theta_2)) \setminus B(x_1,\theta_1)$ and the function $\Upsilon(\theta_1,\theta_2) = \mathcal{L}^n(A_{\theta_1,\theta_2})$. As $\Upsilon(0,\theta) > \mathcal{L}^n(A)$ and $\Upsilon(\theta,0) < \mathcal{L}^n(A)$ for any $\theta > 0$, there exists $t \in]0;1[$ depending on θ such that $\Upsilon(t\theta,(1-t)\theta) = \mathcal{L}^n(A)$ and we set $A_\theta := A_{t\theta,(1-t)\theta}$. By construction, A_θ and $\Omega \setminus A_\theta$ have nonempty interior. The previous result gives the existence of $(p_{\theta,k})_k \subset \mathrm{BV}(\Omega;\{0;1\})$ which weakly* converges to $p_\theta = \mathbf{1}_{A_\theta}$ in $\mathrm{BV}(\Omega)$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_{\theta,k},\mathbf{M}) \leq F(p_\theta,\mathbf{M}). \tag{5.21}$$

Moreover, $\mathcal{L}^n(A \vartriangle A_\theta)$ tends to 0 as $\theta \to 0^+$, and, using

$$\int_{S_{p_\theta}} \langle \mathbf{M}\boldsymbol{\nu}_{p_\theta},\boldsymbol{\nu}_{p_\theta}\rangle^{1/2} \leq \int_{S_p} \langle \mathbf{M}\boldsymbol{\nu}_{p_\theta},\boldsymbol{\nu}_{p_\theta}\rangle^{1/2} + \sqrt{2}\mathcal{H}^{n-1}(\partial B(x_1,\theta_1)) \cup \partial B(x_2,\theta_2)),$$

we get

$$\limsup_{\theta\to 0^+} F(p_\theta,\mathbf{M}) \leq F(p,\mathbf{M}).$$

According to (5.21), with a diagonal extraction there exists a sequence $(p_k)_k$ which weakly* converges to p such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{M}) \leq F(p,\mathbf{M}).$$

Step 3. Assume that $p \in \mathrm{BV}(\Omega;\{0;1\})$ and $\mathbf{M} \in W^{1,r}(\Omega;\mathcal{G})$. Let $(\mathbf{M}_l)_l$ be a sequence as in Proposition 3.2. Since $\mathbf{M}_l \in \mathcal{C}^\infty \cap W^{1,r}_u(\Omega)$, one can apply Step 2 of the proof, which gives

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{M}_l) \leq F(p,\mathbf{M}_l).$$

The same arguments as for (4.2) give that

$$|F(p,\mathbf{M}_l) - F(p,\mathbf{M})| \leq \|\mathbf{M}_l - \mathbf{M}\|_{L^\infty} \frac{\mathcal{H}^{n-1}(S_p)}{2}.$$

So, we deduce that $(F(p,\mathbf{M}_l))_l$ converges to $F(p,\mathbf{M})$. With a diagonal extraction, we can conclude that there exists $(p_k,\mathbf{M}_k)_k \subset \mathcal{Y}$ which converges for the topology $\mathcal{T}$ to $(p,\mathbf{M})$ such that

$$\limsup_{k\to\infty} F_{\varepsilon_k}(p_k,\mathbf{M}_k) \leq F(p,\mathbf{M}).$$

Since $(p_k)_k$ converges pointwise to p it follows that

$$\left(\int_\Omega (p_k - g)^2 \mathrm{d}x\right)_k$$

converges to $\int_\Omega (p-g)^2 \mathrm{d}x$. Moreover, by construction $(\|\mathbf{M}_k\|_{W^{1,r}})_k$ converges to $\|\mathbf{M}\|_{W^{1,r}}$ and we conclude that

$$\limsup_{k\to\infty} E_{\varepsilon_k}(p_k,\mathbf{M}_k) \leq E(p,\mathbf{M}).$$

□

6 Conclusion

We have introduced a model for the detection of thin tubes, we have proved existence of solutions and we have proved an approximation result for this energy suitable in the sense of Γ-convergence. In a forthcoming paper, we will use the approximate problem for numerical experiments. On the other hand, the main hypothesis we did in this paper is the bimodality of the histogram, which is quite restrictive. If this assumption is not ensured the previous model is not valid any longer and has to be modified. We will set a more general formulation that perfoms a similar segmentation without the binary constraint. Roughly speaking, we look for a pair $(f,\mathbf{M})$ where f is a function which is not necessarily binary. The corresponding energy to be minimized will be:

$$\int_\Omega (f-g)^2 + \beta \int_{S_f} \langle \mathbf{M}\boldsymbol{\nu}_f,\boldsymbol{\nu}_f\rangle^{1/2}\, \mathrm{d}\mathcal{H}^{n-1} + \gamma\|\mathbf{M}\|_{W^{1,r}(\Omega)} + \rho \int_{\Omega\setminus S_f} |\nabla f|^2. \tag{6.1}$$

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Chapitre 3

Un modèle dans le cas général

3.1 Résumé

Ce troisième chapitre est la suite logique des deux précédents. Le but est de supprimer l'hypothèse de binarité des deux étapes précédentes et de proposer un modèle dans le cas général. Pour cela, nous organisons notre travail en trois étapes.

1. **Introduction du modèle.** L'énergie de Mumford-Shah associée à g est définie par

$$\mathcal{E}(u,K) = \int_\Omega (u-g)^2 \mathrm{d}x + \beta \mathcal{H}^{n-1}(K) + \gamma \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}x,$$

Pour les raisons développées dans le deuxième chapitre, nous introduisons $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ un champ de matrices symétriques définies positives. Afin de détecter les fins tubes de l'image, la matrice $\mathbf{M}(x)$ admet une valeur propre dominante en tout point x du tube. Le nouveau modèle de Mumford-Shah anisotrope associé à $\mathbf{M}$ est donc

$$\mathcal{E}(u,K) = \int_\Omega (u-g)^2 \mathrm{d}x + \beta \int_K \langle \mathbf{M}\nu, \nu \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \gamma \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}x,$$

où ν est un vecteur unitaire et normal à K. Afin de démontrer que cette énergie admet un minimum (Théorème 1.2.), nous introduisons une formulation relaxée dans l'espace $\mathrm{SBV}(\Omega)$ et nous utilisons le théorème de compacité d'Ambrosio [Amb89]. Il s'agit ensuite de démontrer qu'une solution relaxée est une solution du problème initial. Le point

crucial est de démontrer que si $u \in \mathrm{SBV}(\Omega)$ est un minimiseur alors son ensemble des sauts est essentiellement fermé $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$. Pour cela, nous utilisons un résultat qui généralise le lemme de décroissance de De Giorgi-Carriero-Leaci à une classe de fonctions plus large qui inclue le cas des minimiseurs de notre fonctionnelle [BL13].

Ce travail fait l'objet de l'article [Vic15c] : *An Anisotropic Mumford-Shah Model*, présenté à la fin de ce chapitre.

2. **Un résultat de théorie géométrique de la mesure.** Afin de réaliser une approximation par Γ-convergence que nous présenterons dans la dernière partie de ce chapitre, il est nécessaire d'établir un résultat d'approximation de $\int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$ par rapport à la mesure de Lebesgue. Plus précisément, dans le cas où elle existe, le contenu de Minkowski d'un ensemble S est défini par la limite suivante

$$\mathcal{M}(S) = \lim_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}(x, S) < \rho\})}{2\rho}.$$

Si S est fermé et $(n-1)$-rectifiable, alors on a, dans [Fed69] (Théorème 3.2.39), le résultat suivant bien connu

$$\mathcal{M}(S) = \mathcal{H}^{n-1}(S).$$

Le but est d'établir un résultat équivalent dans le cas où $\mathcal{H}^{n-1}(S)$ est remplacé par $\int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$. Pour cela, nous introduisons le contenu de Minkowski anisotrope suivant

$$\mathcal{M}_{\mathbf{M}}(S) = \lim_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_\phi(x, S) < \rho\})}{2\rho}$$

où dist_ϕ est la distance intégrée définie par

$$\mathrm{dist}_\phi(x, S) = \inf \left\{ \int_0^1 \langle \mathbf{M}^{-1}(\gamma)\dot{\gamma}, \dot{\gamma}\rangle^{\frac{1}{2}} \mathrm{d}t \colon \begin{array}{l} \gamma \in W^{1,1}([0;1];\Omega), \\ \gamma(0) = x, \gamma(1) \in S \end{array} \right\}.$$

Nous démontrons alors que si S est fermé, $(n-1)$-rectifiable et $\mathbf{M}$ vérifie

$$\exists \alpha > 0, \exists \theta > 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \theta |x - y|^\alpha,$$

alors

$$\mathcal{M}_{\mathbf{M}}(S) = \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}.$$

Ce travail fait l'objet de l'article [Vic15b] : *Anisotropic Minkowski Content of a Surface*, présenté à la fin de ce chapitre.

3. **Un résultat de Γ-convergence.** Dans cette dernière partie, nous établissons une approximation de la fonctionnelle

$$E(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \beta \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1} + \gamma \int_\Omega |\nabla u|^2 \mathrm{d}x$$

   où $u \in \mathrm{SBV}(\Omega)$ et J_u l'ensemble des sauts de u, par une famille de fonctionnelles $(E_\varepsilon)_\varepsilon$ définies par

$$E_\varepsilon(u,z) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x.$$

   La fonction z prend ses valeurs dans $[0;1]$ et joue le rôle de contrôle du gradient de u. À l'aide du résultat obtenu dans l'étape précédente, nous démontrons que la famille $(E_\varepsilon)_\varepsilon$ Γ-converge vers E lorsque $\varepsilon \to 0^+$.

Ce travail fait l'objet de l'article [Vic15d] : *Approximation of an Anisotropic Mumford-Shah Functional with* Γ*-convergence*, présenté à la fin de ce chapitre.

3.2 Articles [Vic15c][Vic15b][Vic15d]

An Anisotropic Mumford-Shah Model

David Vicente

April 23, 2015

Abstract

We introduce an anisotropic Mumford-Shah functional in dimension $n = 2, 3$. To detect the thin tubular structures of an image, the classical Hausdorff measure in the original model is replaced by an anisotropic surface measure depending on a riemannian metric $\mathbf{M}$. We then consider a relaxation of this energy in the set of SBV functions and we prove that the minimizing problem admits solution under suitable conditions. We also prove that a relaxed solution provides in fact a regular solution to the initial problem.

Introduction

This work is a contribution to the problem of detection of thin structures, namely tubes, in a digital image with dimension $n = 2$ or $n = 3$. In a previous work [1], we have introduced an energy in the binary context. More precisely, we assumed that the image histogram was bimodal. In this paper, we remove this assumption and generalize our previous results. To solve this problem, we modify the so-called Mumford-Shah model [2] by introducing a geometric prior which favors tubes. The domain of the image is an open and bounded set $\Omega \subset \mathbb{R}^n$, $g : \Omega \to \mathbb{R}$ is a given image with normalized gray level in $[0; 1]$ and the well-known Mumford-Shah energy associated to this image is defined as

$$E(u, K) = \int_{\Omega\setminus K} (u - g)^2 \mathrm{d}x + \int_{\Omega\setminus K} |\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(K),$$

where K a compact subset of Ω and $u \in W^{1,2}(\Omega \setminus K)$. To favors the detection of sets with tubular geometry, we introduce a riemannian metric $\mathbf{M}$ which contains the local orientation of the tubes. Formally, $\mathbf{M}$ is a function defined on Ω with values in the set $\mathcal{S}_n^+(\mathbb{R})$ of symmetric positive definite matrices. To give an idea, $\mathbf{M}$ may take the form

$$\forall x \in \Omega,\ \forall \mathbf{v} \in \mathbf{R}^n, \quad \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle = |\mathbf{v}|^2 + \langle \mathbf{c}(x), \mathbf{v}\rangle^2$$

where $\mathbf{c} : \Omega \to \mathbf{S}^{n-1}$ is an unitary vector field which locally gives the direction of the tube. If we assume that K is a $\mathcal{C}^1$-hypersurface, we may replace $\mathcal{H}^{n-1}(K)$ by its associated anisotropic version $\int_K \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$, where $\nu : K \to \mathbb{S}^{n-1}$ an unitary, normal vector to K. So, to minimize this term, ν has to be orthogonal to $\mathbf{c}$ and then it favors images for which K is tangent to the field $\mathbf{c}$. The Mumford-Shah energy associated to this metric can be defined as

$$E_{\mathbf{M}}(u, K) = \int_{\Omega\setminus K} (u - g)^2 \mathrm{d}x + \int_{\Omega\setminus K} |\nabla u|^2 \mathrm{d}x + \int_K \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \tag{0.1}$$

where K is a compact $\mathcal{C}^1$-hypersurface, $\nu : K \to \mathbb{S}^{n-1}$ an unitary, normal vector to K, and $u \in W^{1,2}(\Omega \setminus K)$.

In section 1, we introduce a relaxed formulation of this problem and prove that it admits a solution. In section 2, we show a regularity result and prove that it provides a solution to the initial unrelaxed problem. In section 3, we present various techniques to construct the metric $\mathbf{M}$.

1 Relaxed problem

In order to prove that the minimization of $E_{\mathbf{M}}$ is a well-posed problem, we introduce a relaxed formulation and prove that the new relaxed problem admits a solution.

1.1 Functional framework

The following definitions and results are taken from [4], chapters 3 and 4. A function $u \in L^1(\Omega)$ is said with *bounded variation*, denoted $u \in BV(\Omega)$, if its derivative, in the sense of the distribution, is a Radon measure.

We are interested by the property for this space to allow functions with *jump* discontinuities. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^n$ and introduce

$$\begin{cases} B_r^+(x,\nu) = \{y \in B_r(x) \colon \langle y - x, \nu \rangle > 0\}, \\ B_r^-(x,\nu) = \{y \in B_r(x) \colon \langle y - x, \nu \rangle < 0\}, \end{cases}$$

for the two half balls contained in the ball $B_r(x) \subset \mathbb{R}^n$ determined by $\nu \in \mathbb{S}^{n-1}$.

Definition 1.1. *Let $u \in L^1(\Omega)$ and $x \in \Omega$. We say that x is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and*

$$\lim_{r \to 0^+} \fint_{B_r^+(x,\nu)} |u(y) - a| \, dy = 0, \qquad \lim_{r \to 0^+} \fint_{B_r^-(x,\nu)} |u(y) - b| \, dy = 0.$$

The set of approximate jump points is denoted by J_u. The triplet (a, b, ν), uniquely determined up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

The set J_u inherits the following structure theorem.

Theorem 1.1. *Let u be a given function in $BV(\Omega)$. Then, J_u is countably $(n-1)$-rectifiable. There exists a countable family $(K_i)_i$ of compact $\mathcal{C}^1$-hypersurfaces such that $J_u = N \cup (\bigcup_i K_i)$, where $\mathcal{H}^{n-1}(N) = 0$.*

We say that $u \in BV(\Omega)$ is a *special function with bounded variation* and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative is zero, we obtain:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure $\mathcal{L}^n$, ν_u the normal of the jump set J_u and $\mathcal{H}^{n-1} \llcorner J_u$ the restriction of the Hausdorff measure to the jump set.

1.2 Existence result

For $u \in SBV(\Omega)$, replacing K by J_u in (0.1), we define the relaxed energy by

$$\tilde{E}_{\mathbf{M}}(u) = \int_\Omega (u - g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \tag{1.1}$$

where ∇u, J_u and ν_u are defined in the sense of $SBV(\Omega)$. We denote by $(\tilde{\mathcal{P}}_{\mathbf{M}})$ the relaxed problem

$$(\tilde{\mathcal{P}}_{\mathbf{M}}): \quad \min\{\tilde{E}_{\mathbf{M}}(u) \colon u \in SBV(\Omega)\}.$$

We introduce the following constraints on $\mathbf{M}$:

i) ellipticity:

$$(H_1): \qquad \exists\lambda > 0, \exists\Lambda > 0, \forall(x,v) \in \Omega \times \mathbb{R}^n, \quad \lambda|v|^2 \leq \langle \mathbf{M}(x)v, v\rangle \leq \Lambda|v|^2,$$

ii) Hölder-regularity:

$$(H_2): \qquad \exists\alpha > 0, \exists C \geq 0, \forall(x,y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq C|x-y|^\alpha.$$

Condition (H_1) is equivalent to the inclusion of $\mathbf{M}$ spectrum in $[\lambda;\Lambda]$. If $\mathbf{M} \in W^{1,p}(\Omega)$ and $p > n$ then, according to Sobolev embedding theorem (see [3], chapter 5), condition (H_2) is satisfied with $\alpha = 1 - \frac{n}{p}$.

In this section, we will prove the following result.

Theorem 1.2. *Let $\tilde{E}_\mathbf{M}$ be defined as in (1.1) and $\mathbf{M}$ a metric which satisfies (H_1) and (H_2). Then, the problem $(\tilde{\mathcal{P}}_\mathbf{M})$ admits at least one solution.*

To prove this result, we will use the *direct method* of calculus of variation. The key tools are Theorem 4.8. (compactness) Theorem 4.7. (lower semi-continuity) of [4] in the context of a constant and homogeneous metric (that is $\mathbf{M} \equiv \mathrm{Id}$). Our result is a generalization: $\mathbf{M}$ is not necessary the identity matrix (anisotropy) and may depend on $x \in \Omega$ (non homogeneity).

In the sequel we assume that the hypothesis of Theorem 1.2 are satisfied. We use the weak*-convergence which is defined in [4], definition 3.11.

Lemma 1.1 (Compactness)**.** *Let $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$ be such that $\left(\tilde{E}_\mathbf{M}(u_k)\right)_k$ is bounded. Then, there exists a sequence weakly* convergent to $u \in SBV(\Omega)$.*

Proof. According to ellipticity condition (H_1), we have

$$\int_\Omega |\nabla u_k|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k}) \leq \max\left\{1; \lambda^{-\frac{1}{2}}\right\} \tilde{E}_\mathbf{M}(u_k). \tag{1.2}$$

In [4] (Theorem 4.8.), it is proved that the boudedness of the left hand side of (1.2) ensures the existence of a subsequence $(u_k)_k$ weakly* converging in $BV(\Omega)$ to $u \in \mathrm{SBV}(\Omega)$.

□

Lemma 1.2 (Lower semicontinuity)**.** *Let $(u_k)_k \subset SBV(\Omega)$ be a weakly* convergent sequence to $u \in SBV(\Omega)$. Then, we have*

$$\tilde{E}_\mathbf{M}(u) \leq \liminf_{k\to\infty} \tilde{E}_\mathbf{M}(u_k).$$

Proof. As $(u_k)_k$ weakly* converges to u then it converges in $L^1(\Omega)$ and

$$\lim_{k\to\infty} \int_\Omega (u_k - g)^2 \mathrm{d}x = \int_\Omega (u-g)^2 \mathrm{d}x. \tag{1.3}$$

We may assume that $(\tilde{E}_\mathbf{M}(u_k))_k$ is bounded, otherwise the result is ensured. So, according to inequality (1.2), $\int_\Omega |\nabla u_k|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_k})$ is bounded with respect to k. With [4] (Theorem 4.7.), it implies that

$$\int_\Omega |\nabla u|^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 \mathrm{d}x, \tag{1.4}$$

$$\mathcal{H}^{n-1}(J_u) \leq \liminf_{k\to\infty} \mathcal{H}^{n-1}(J_{u_k}). \tag{1.5}$$

According to (1.3), (1.4) and (1.5), it is sufficient to prove that

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}. \tag{1.6}$$

This result is proved in [4] (Theorem 5.2.) for an homogeneous and fixed media (if the metric $\mathbf{M}$ does not depend on $x \in \Omega$). In order to overpass this constraint, we introduce a piecewise constant approximation. Then, we apply the result of [4] for each piece of the approximation.

Let $\eta > 0$ be arbitrary small and $\mathcal{A}$ be a finite partition of Ω, such that, for any $A \in \mathcal{A}$, $\mathrm{diam}(A) < \eta$. For each set $A \in \mathcal{A}$, we fix one point $x_A \in A$. We denote $\mathbf{M}^{\mathcal{A}}$ the metric such as its restriction on A is equal to $\mathbf{M}(x_A)$. Moreover, for any vector of the canonical basis $e_i \in \mathbb{S}^{n-1}$, we denote

$$\begin{cases} \Pi^i_t = \{x \in \Omega \colon \langle x, e_i\rangle = t\}, \\ N^i_t = \{t \in \mathbb{R} \colon \mathcal{H}^{n-1}(J_u \cap \Pi^i_t) > 0\} \cup \{t \in \mathbb{R} \colon \exists k \in \mathbb{N}, \mathcal{H}^{n-1}(J_{u_k} \cap \Pi^i_t) > 0\}. \end{cases}$$

As $\mathcal{H}^{n-1}(J_u) < \infty$ (1.5) and $\mathcal{H}^{n-1}(J_{u_k}) < \infty$, then N^i_t is at most countable. So, for any fixed $\eta > 0$, there exists a finite partition $\mathcal{A}$ of Ω such that any $A \in \mathcal{A}$ satisfies

$$\begin{cases} \partial A \subset \bigcup_{i,j} \Pi^i_{t_{i,j}}, \\ \forall (x,y) \in A^2,\ |x-y| \leq \eta, \\ \mathcal{H}^{n-1}(J_u \cap \partial A) = 0, \\ \forall k \in \mathbb{N},\ \mathcal{H}^{n-1}(J_{u_k} \cap \partial A) = 0. \end{cases} \tag{1.7}$$

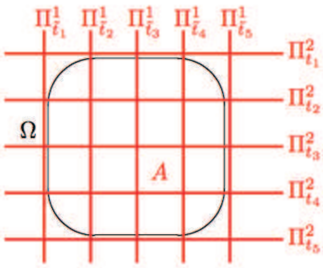

Figure 1.1: Construction of $\mathcal{A}$

We will estimate the limits of the following integral.

$$\begin{aligned} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} &= \int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \right) \mathrm{d}\mathcal{H}^{n-1} \\ &\quad + \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}. \end{aligned} \tag{1.8}$$

Claim 1: The sequence

$$\int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \right) d\mathcal{H}^{n-1}$$

converges to 0 *uniformly with respect to* $k \in \mathbb{N}$ *when* η *converges to* 0^+.

Let be $A \in \mathcal{A}$, $x \in A$ and estimate

$$\begin{aligned} \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} &= \langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}, \\ &= \frac{\langle (\mathbf{M}(x) - \mathbf{M}(x_A))\nu_{u_k}, \nu_{u_k}\rangle}{\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}}. \end{aligned} \tag{1.9}$$

According to ellipticity condition (H_1), it yields

$$\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} + \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \geq 2\lambda^{\frac{1}{2}}, \quad \text{for } \mathcal{H}^{n-1}\llcorner J_{u_k} - \text{a.e. } x \in A. \tag{1.10}$$

According to regularity assumption (H_2), there exist constants $C > 0$ and $\alpha > 0$ such that

$$\forall x \in A, \quad \|\mathbf{M}(x) - \mathbf{M}(x_A)\| \leq C|x - x_A|^\alpha \leq C\eta^\alpha. \tag{1.11}$$

So, (1.9), (1.10) and (1.11) give

$$\left|\langle \mathbf{M}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}(x)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}\right| \leq \frac{C\eta^\alpha}{2\lambda^{\frac{1}{2}}}, \quad \text{for } \mathcal{H}^{n-1}\llcorner J_{u_k} - \text{a.e. } x \in A.$$

As $\mathcal{A}$ is a partition of Ω, we have

$$\left|\int_{J_{u_k}} \left(\langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}\right) \mathrm{d}\mathcal{H}^{n-1}\right| \leq \frac{C\eta^\alpha \mathcal{H}^{n-1}(J_{u_k})}{2\lambda^{\frac{1}{2}}}.$$

As ellipticity condition gives $\mathcal{H}^{n-1}(J_{u_k}) \leq \lambda^{-\frac{1}{2}}\tilde{E}_{\mathbf{M}}(u_k)$, then $(\mathcal{H}^{n-1}(J_{u_k}))_k$ is a bounded sequence and it concludes the proof of *Claim 1*.

Claim 2: We have the following result

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u\rangle^{\frac{1}{2}}\, d\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}\, d\mathcal{H}^{n-1}.$$

We denote by $\overset{\circ}{A}$ the interior of the set A. According to [4], Theorem 5.2., we have

$$\int_{J_u \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k} \cap \overset{\circ}{A}} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

According to (1.7), the energy on the boundaries is null. It gives

$$\int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

So,

$$\begin{aligned}
\sum_{A\in\mathcal{A}} \int_{J_u \cap A} \langle \mathbf{M}(x_A)\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} &\leq \sum_{A\in\mathcal{A}} \liminf_{k\to\infty} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\
&\leq \liminf_{k\to\infty} \sum_{A\in\mathcal{A}} \int_{J_{u_k} \cap A} \langle \mathbf{M}(x_A)\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

As $\mathcal{A}$ is a partition of Ω, this conclude the proof of *Claim 2*:

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

Now, let $\delta > 0$ be an arbitrary small number. According to *Claim 1*, there exists $\eta > 0$ and a partition $\mathcal{A}$ defined as above which satisfies

$$\limsup_{k\to\infty} \left|\int_{J_{u_k}} \left(\langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}}\right) \mathrm{d}\mathcal{H}^{n-1}\right| \leq \delta. \tag{1.12}$$

According to decomposition (1.8), (1.12) and *Claim 2*, we have

$$\int_{J_u} \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \delta + \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

As for *Claim 2*, we have

$$\left| \int_{J_u} \left(\langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} - \langle \mathbf{M}^{\mathcal{A}}\nu_u, \nu_u \rangle^{\frac{1}{2}} \right) \mathrm{d}\mathcal{H}^{n-1} \right| \leq \frac{C\eta^{\alpha}\mathcal{H}^{n-1}(J_u)}{2\lambda^{\frac{1}{2}}}.$$

We may conclude

$$\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \leq \liminf_{k\to\infty} \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

□

Recall that we have the following chain rule for $SBV(\Omega)$.

Theorem 1.3. *Let $u \in SBV(\Omega)$ and let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then, $v = \varphi \circ u$ belongs to $SBV(\Omega)$ and*

$$Dv = \varphi'(u)\nabla u\mathcal{L}^n + (\varphi(u^+) - \varphi(u^-))\nu_u\mathcal{H}^{n-1} \llcorner J_u. \tag{1.13}$$

This result is a straightforward consequence of Theorem 3.99 in [4]. It is the key tool for the proof of Theorem 1.2 that follows.

Proof. We denote by $(u_k)_k \subset SBV(\Omega)$ a minimizing sequence for $\tilde{E}_{\mathbf{M}}$. As we assumed that $g \in L^\infty(\Omega)$, we may introduce

$$\forall t \in \mathbb{R}, \quad \varphi(t) = \begin{cases} -\|g\|_{L^\infty(\Omega)} & \text{if } t \leq -\|g\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \leq \|g\|_{L^\infty(\Omega)}, \\ \|g\|_{L^\infty(\Omega)} & \text{if } t \geq \|g\|_{L^\infty(\Omega)}. \end{cases}$$

We denote $v_k = \varphi \circ u_k$. As the function φ is 1-Lipshitz, we may apply Theorem 1.3, then $v_k \in SBV(\Omega) \cap L^\infty(\Omega)$. According to the decomposition (1.13), we have

$$\forall k, \quad \tilde{E}_{\mathbf{M}}(v_k) \leq \tilde{E}_{\mathbf{M}}(u_k),$$

so $(v_k)_k$ is a minimizing sequence for $\tilde{E}_{\mathbf{M}}$. According to Theorem 1.1, there exists $v \in SBV(\Omega)$ and a subsequence, still denoted $(v_k)_k$ weakly* convergent to v. With Theorem 1.2, we have $\tilde{E}_{\mathbf{M}}(v) \leq \liminf \tilde{E}_{\mathbf{M}}(v_k)$. So, v is a minimizer of $\tilde{E}_{\mathbf{M}}$.

□

2 Regularity result

An important question is to check if a SBV minimizer of the relaxed problem $(\tilde{\mathcal{P}}_{\mathbf{M}})$ is a "classical" one i.e. its jump set is closed and so the function is locally smooth in the complement of the jump set. A positive answer was given by De Giorgi, Carriero and Leaci in [5] for the Mumford-Shah functional. We generalize this result and prove that a minimizer of the relaxed problem $(\tilde{\mathcal{P}}_{\mathbf{M}})$, whose existence is proved in section 2, provides a minimizer of the original problem

$$(\mathcal{P}_{\mathbf{M}}): \quad \min\{E_{\mathbf{M}}(u, K) \colon K \subset \Omega \text{ is a compact } \mathcal{C}^1 \text{ hypersurface }, u \in W^{1,2}(\Omega \setminus K)\}.$$

We give the definition of a *local almost-quasi minimizer of a free discontinuity problem* and a regularity result for its jump set which is proved in [6].

Definition 2.1. *We say that $w \in SBV(U)$ is an almost-quasi minimizer of a free discontinuity problem, if there exists $\varLambda \geq 1$, $\alpha > 0$ and $c_\alpha \geq 0$ such that*

$$v \in SBV(U), \quad x \in U, \quad \overline{B_r}(x) \subset U, \quad [w \neq v] \subset B_r(x) \qquad \Rightarrow$$
$$\int_{B_r(x)} |\nabla w|^2 dx + \mathcal{H}^{n-1}(J_w \cap \overline{B_r}(x)) \leq \int_{B_r(x)} |\nabla v|^2 dx + \varLambda\mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + c_\alpha r^{n-1+\alpha}. \tag{2.1}$$

Theorem 2.1. *Let u be an almost-quasi minimizer of a free discontinuity problem, then $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$.*

We use this key tool to prove the following.

Theorem 2.2. *Let u be a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$, then $\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$.*

Proof. Let $u \in SBV(\Omega)$ be a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$. For $\beta > 0$, we denote

$$\forall x \in \beta\Omega, \qquad u_\beta(x) = u\left(\frac{x}{\beta}\right), \quad g_\beta(x) = g\left(\frac{x}{\beta}\right).$$

As

$$\mathcal{H}^{n-1}(\overline{J_{u_\beta}} \setminus J_{u_\beta}) = 0 \Rightarrow \mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0$$

then, according to Theorem 2.1, it suffices to prove the following assertion.

Claim : There exists $\beta > 0$ such that $u_\beta \in SBV(\beta\Omega)$ is an almost-quasi minimizer of a free discontinuity problem

With the same argument as in the proof of Theorem 1.2, we have $u \in SBV(\Omega) \cap L^\infty(\Omega)$. As u is a minimizer of $(\tilde{\mathcal{P}}_{\mathbf{M}})$, then u_β is a minimizer of the rescaled problem

$$(\tilde{\mathcal{P}}^\beta_{\mathbf{M}}) : \quad \min\left\{E^\beta_{\mathbf{M}}(v), v \in SBV(\beta\Omega)\right\},$$

where

$$\tilde{E}^\beta_{\mathbf{M}}(v) = \beta^2 \int_{\beta\Omega} (v - g_\beta)^2 \mathrm{d}x + \int_{\beta\Omega} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

Let be $v \in SBV(\beta\Omega)$, $\quad x \in \beta\Omega$, $\quad \overline{B_r}(x) \subset \beta\Omega$, $\quad [u_\beta \neq v] \subset B_r(x)$ and $\tilde{v} = \varphi \circ v$, where φ is introduced in the proof of Theorem 1.2. As u_β is a minimizer of $(\tilde{\mathcal{P}}^\beta_{\mathbf{M}})$ then $E^\beta_{\mathbf{M}}(u_\beta) \leq E^\beta_{\mathbf{M}}(\tilde{v})$ and it implies

$$\begin{aligned} &\int_{B_r(x)} |\nabla u_\beta|^2 \mathrm{d}x + \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \\ \leq &\int_{B_r(x)} |\nabla \tilde{v}|^2 \mathrm{d}x + \beta \int_{J_{\tilde{v}} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + \int_{B_r(x)} (\tilde{v} - g_\beta)^2 \mathrm{d}x. \end{aligned}$$

Then

$$\begin{aligned} &\int_{B_r(x)} |\nabla u_\beta|^2 \mathrm{d}x + \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \\ \leq &\int_{B_r(x)} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_v, \nu_v\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + 4\|g\|^2_{L^\infty(\Omega)} \omega_n r^n, \end{aligned}$$

where $\omega_n = \mathcal{L}^n(B_1(x))$. Now, we set $\beta = \lambda^{-1}$ where λ is the ellipticity coefficient introduced in section 1.1. The left hand side of inequality (H_1) gives

$$\begin{aligned} \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) &= \beta\lambda \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)), \\ &\leq \beta \int_{J_{u_\beta} \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_{u_\beta}, \nu_{u_\beta}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

so

$$\begin{aligned} &\int_{B_r(x)} |\nabla u_\beta|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x)) \\ \leq &\int_{B_r(x)} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v \cap \overline{B_r}(x)} \langle \mathbf{M}\nu_v, \nu_v\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + 4\|g\|^2_{L^\infty(\Omega)} \omega_n r^n. \end{aligned}$$

The right hand side of (H_1) gives

$$\int_{B_r(x)} |\nabla u_\beta|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_{u_\beta} \cap \overline{B_r}(x))$$
$$\leq \int_{B_r(x)} |\nabla v|^2 \mathrm{d}x + \beta\Lambda\mathcal{H}^{n-1}(J_v \cap \overline{B_r}(x)) + 4\|g\|^2_{L^\infty(\Omega)}\omega_n r^n.$$

So, we may conclude that u_β satisfies the definition of an almost quasi-minimizer of a free discontinuity problem and the *Claim* is proved.

□

We deduce from the previous Theorem that a minimizer of the relaxed problem provides a minimizer of the general problem. Moreover, we have

Proposition 2.1. *Let $u \in SBV(\Omega)$ be a minimizer of $(\mathcal{P}_\mathbf{M})$, then $u \in \mathcal{C}^1(\Omega \setminus \overline{J}_u)$.*

Proof. Let $\overline{B}_r(x) \subset \Omega \setminus \overline{J}_u$; then $u \in W^{1,2}(B_r(x))$ and it is a minimizer of the functional

$$\int_{B_r(x)} (v-g)^2 \mathrm{d}x + \int_{B_r(x)} |\nabla v|^2 \mathrm{d}x$$

among the functions v in $u + W_0^{1,2}(B_r(x))$ and then classical regularity results give $u \in \mathcal{C}^1(B_r(x))$. □

3 Construction of M

In the previous sections, we assumed the existence of a riemannian metric $\mathbf{M}$ adapted to the problem of detection of tubes. Moreover, our results are true if ellipticity (H_1) and Holder-regularity (H_2) are satisfied. So, we propose two possible definitions of such a metric which may be used in practice.

3.1 2D Case

We give a definition adapted to dimension 2. For that, we search for an unitary vector field $\mathbf{c} : \Omega \to \mathbb{S}^1$ following the direction of the tubes.

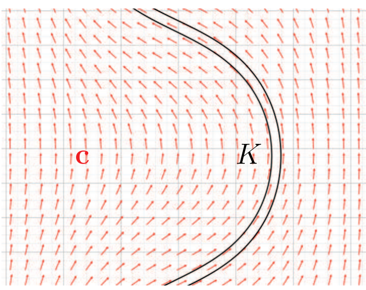

Figure 3.1: Vector field $\mathbf{c}$ along a tube K

We introduce the following functional

$$F(\mathbf{c}) = \int_\Omega \langle Dg, \mathbf{c}\rangle^2 \mathrm{d}x + \int_\Omega |D\mathbf{c}|^p \mathrm{d}x$$

and the following minimization problem

$$(\mathcal{P}_c): \quad \min\{F(\mathbf{c}) : \mathbf{c}(x) \in \mathbb{S}^1 \text{ } a.e. \text{ } x \in \Omega, \, \mathbf{c} \in W^{1,p}(\Omega)\}.$$

If we set $p > 2$ then, Sobolev embedding Theorem ensures that $\mathbf{c}$ is α-Holder regular with $\alpha = 1 - \frac{2}{p}$. It is easy to prove that a solution $\mathbf{c}_0$ of $(\mathcal{P}_c)$ exists and we set

$$\mathbf{M} = \mathrm{Id} + \mu\, {}^t\mathbf{c}_0\mathbf{c}_0,$$

where $\mu > 0$ corresponds to the elongation of the unit ball of $\mathbf{M}(x)$ along the direction $\mathbf{c}(x)$.

3.2 3D Case

In dimension 3, the previous approach is not adapted. In fact, a vector field can avoid *lateraly* a tube without penalizing the regularization term $\int_\Omega |D\mathbf{c}|^p \mathrm{d}x$.

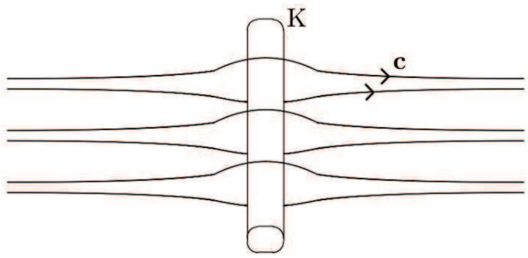

Figure 3.2: Vector field $\mathbf{c}$ avoiding lateraly a tube K

To overpass this problem, we introduce the second order derivative of $\mathbf{H}$ of g and the following minimization problem

$$F(\mathbf{M}) = \int_\Omega \|\mathbf{M} - \mathbf{H}\|^2 \mathrm{d}x + \int_\Omega \|D\mathbf{M}\|^p \mathrm{d}x$$

and the following minimization problem

$$(\mathcal{P}_H): \quad \min\{F(\mathbf{M}) : \mathbf{M} \text{ satisfies } (H_1), \mathbf{M} \in W^{1,p}(\Omega)\}.$$

If we assume that $\mathbf{H} \in L^2(\Omega)$, then it easy to prove that this problem admits a solution $\mathbf{M}_0$. As for the 2D case, we assume that $p > 3$ and Sobolev embedding Theorem ensures that $\mathbf{M}_0$ satisfies (H_2).

Conclusion

We have introduced a new model and we have proved that the associated minimizing problem is well posed. In a forthcoming work, we will introduce an approximation of this problem with Γ-convergence. It allows us to solve the minimizing problem with PDE technics.

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Anisotropic Minkowski Content and Application to almost-quasi minimizers of free discontinuity problems

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June 24, 2015

Abstract

This paper deals with a generalization of a result on Minkowski contents: if a set is closed and $(n-1)$-rectifiable, then its Minkowski content coincides with its $(n-1)$-dimensional Hausdorff measure. In our case, the Hausdorff measure is replaced by an anisotropic measure depending on the location and on the orientation of the set. We introduce an adapted Minkowski content and we prove that, under the same hypothesis of closure and rectifiability, the anisotropic Hausdorff measure and the anisotropic Minkowski content coincide. Then, we apply this result to an almost quasi minimizer of a free boundary problem and we prove that the anisotropic Minkowski content of its jump set is equal to its anisotropic Hausdorff measure.

1 Introduction

In this paper we are interested in anisotropic measures defined on $(n-1)$-rectifiable surfaces $S \subset \mathbb{R}^n$ by

$$\mathcal{S}_{\mathbf{M}}(S) = \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1},$$

where $\mathbf{M} : \mathbb{R}^n \to \mathrm{S}_n^+(\mathbb{R})$ is a field of symmetric definite positive matrices, $\boldsymbol{\nu}$ is a unitary vector orthogonal to S and $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure. We recall that for $S \subset \mathbb{R}^n$, the $(n-1)$-dimensional upper and lower Minkowski contents are defined by

$$\mathcal{M}^\star(S) = \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \colon \mathrm{dist}(x,S) < \rho\})}{2\rho}, \quad \mathcal{M}_\star(S) = \liminf_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \colon \mathrm{dist}(x,S) < \rho\})}{2\rho}.$$

In case these upper and lower Minkowski contents are equal, their common value is called the $(n-1)$-dimensional Minkowski content $\mathcal{M}(S)$. In [1], the following result is given.

Theorem 1.1 (Federer, Theorem 3.2.39). *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$, then*

$$\mathcal{M}(S) = \mathcal{H}^{n-1}(S).$$

As $\mathcal{H}^{n-1}(S) = \mathcal{S}_{\mathrm{Id}}(S)$, we generalize this result for the case where $\mathbf{M}$ is not necessary equal to the identity. For that, we introduce the following Riemannian metric

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}} \tag{1.1}$$

and the integrated distance associated to ϕ is

$$\begin{aligned} \forall (x,y) \in \Omega^2, \quad \mathrm{dist}_\phi(x,y) &= \inf \left\{ \int_0^1 \phi(\gamma, \dot{\gamma}) \mathrm{d}t \colon \begin{array}{c} \gamma \in W^{1,1}([0;1];\Omega), \\ \gamma(0) = x, \gamma(1) = y \end{array} \right\}, \\ \forall x \in \Omega, \quad \mathrm{dist}_\phi(x,S) &= \inf \{ \mathrm{dist}_\phi(x,y) \colon y \in S \}, \end{aligned} \tag{1.2}$$

where $\dot{\gamma} = \frac{d\gamma}{dt}$. We define the associated anisotropic Minkowski $(n-1)$-dimensional upper and lower content as the limits

$$\begin{aligned} \mathcal{M}^\star_{\mathbf{M}}(S) &= \limsup_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_\phi(x,S) < \rho\})}{2\rho}, \\ \mathcal{M}_{\star\mathbf{M}}(S) &= \liminf_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_\phi(x,S) < \rho\})}{2\rho}. \end{aligned}$$

In case they are equal, we call their common value the $(n-1)$-dimensional anisotropic Minkowski content $\mathcal{M}_{\mathbf{M}}(S)$. The main result we prove in this paper is the following.

Theorem 1.2. *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$ and $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ satisfies Hölder-regularity condition:*

$$(H): \qquad \exists \alpha > 0, \exists \theta > 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \theta |x-y|^{\alpha},$$

then we have

$$\mathcal{M}_{\mathbf{M}}(S) = \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1},$$

where $\boldsymbol{\nu}$ is an unitary and normal vector to S.

If $\mathbf{M}$ is defined on a bounded domain Ω, then Sobolev embedding Theorem ensures that if $\mathbf{M} \in \mathrm{W}^{1,p}(\Omega; \mathrm{S}_n^+(\mathbb{R}))$ and $p > n$, then $\mathbf{M}$ satisfies (H).

A comparable result is given in [2] (Theorem 6.1) for $S = \partial E$ and E is a set of finite perimeter. In [3], the study focuses on anisotropic outer Minkowski content for the same class of sets. Motivated by the Mumford-Shah anisotropic model introduced in [5], we are interested in the extension of those results to the larger class of $(n-1)$-rectifiable sets (which includes $S = \partial E$, E with finite perimeter).

In section 2, we motivate our result and give an heuristical approach for introducing ϕ and $\mathcal{M}_{\mathbf{M}}$. In section 1.2 we prove the main result in two steps: first in 3.1, we assume that $\mathbf{M}$ does not depend on $x \in \Omega$ and give a proof of our result under this assumption; then, in section 3.2, we generalize the proof in the inhomogeneous case. In section 4, we apply this result to the case where S is given by the jump set a-of an almost-quasi-minimizer.

2 Origin of the problem and heuristic

We adopt the notations:

- Ω an open and bounded subset of $\mathbb{R}^n$,
- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \in \mathbb{R}$ for the canonical scalar product of $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$,
- $\bigwedge_{i=1}^{n-1} \mathbf{v}_i \in \mathbb{R}^n$ for the canonical vectorial product of $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1} \in \mathbb{R}^n$,
- $|\mathbf{v}|$ for the euclidean norm of $\mathbf{v} \in \mathbb{R}^n$,
- $\|\mathbf{M}\|$ for an induced norm of $\mathbf{M} \in \mathrm{M}_n(\mathbb{R})$,
- $\mathrm{S}_n^+(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of symmetric definite positive matrices,
- $\mathrm{GL}_n(\mathbb{R}) \subset \mathrm{M}_n(\mathbb{R})$ for the subset of invertible matrices,
- $\mathrm{O}_n(\mathbb{R}) \subset \mathrm{GL}_n(\mathbb{R})$ for the subgroup of orthogonal matrices,
- $\mathcal{B}(\Omega)$ the class of Borelian subsets of Ω,
- $\mathcal{L}^n$ for the Lebesgue measure in $\mathbb{R}^n$,
- $\mathcal{H}^{n-1}$ for the $(n-1)$-dimensional Hausdorff measure,
- dist for the euclidean distance in $\mathbb{R}^n$.

2.1 Motivation

This work is a part of a contribution the problem of detection of filaments surrounded by noise in a digital image. In [4] we introduced a model under the assumption of bimodality of the histogram of the image. To remove this assumption, in [5] we have introduced an anisotropic version of the so-called Mumford-Shah model. For the need of numerical implementation, we want to perform a Γ-convergence approximation as it has been done in [6] for the original Mumford-Shah model. To do that, we need to establish a link between Hausdorff measure and Minkowski content in the anisotropic setting.

We set $\Omega \subset \mathbb{R}^n$ an open and bounded domain, with $n \in \{2, 3\}$, and $g : \Omega \to \mathbb{R}$ the image. In [4], we have considered an unitary vector field $\mathbf{c} : \Omega \to \mathbb{S}^{n-1}$ tangent to the filaments (see Figure 2.1).

To favor the detection of thin tubes, for any $x \in \Omega$ we have set

$$\mathbf{M}(x) = \mathrm{Id}_n + \mu \, \mathbf{c}(x)(\mathbf{c}(x))^t,$$

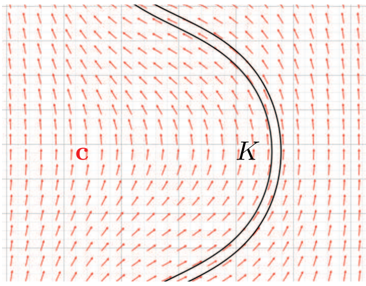

Figure 2.1: Vector field $\mathbf{c}$ along a tube K

where $\mu > 0$ is a parameter adapted to the elongation of the tubes. Then, $\mathbf{M}$ is a field of symmetric definite positive matrices $\mathrm{S}_n^+(\mathbb{R})$ which satisfies

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle = |\mathbf{v}|^2 + \mu \langle \mathbf{c}(x), \mathbf{v}\rangle^2.$$

Roughly speaking, $\mathbf{M}$ contains the local anisotropy of the image and is adapted to the detection of thin tubes (see [4]). We recall the so called Mumford-Shah model

$$E(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \mathcal{H}^{n-1}(J_u),$$

with $u \in SBV(\Omega)$ and J_u is the jump set of u. In [5], we have introduced an anisotropic version of the Mumford-Shah model as follows

$$E_{\mathbf{M}}(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\boldsymbol{\nu}_u, \boldsymbol{\nu}_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \tag{2.1}$$

where $\boldsymbol{\nu}_u$ is an unitary and normal vector to J_u. We remark that if $\mu = 0$, then $\mathbf{M} = \mathrm{Id}_n$ and $E_{\mathbf{M}} = E$. So, our model is a generalization of Mumford-Shah one. We want to perform a Γ-convergence approximation in the same spirit as [6]. In this paper, for the *upper inequality* of Γ-convergence, the authors used Theorem 1.1. In our anisotropic setting, we need to establish the link between $\int_{J_u} \langle \mathbf{M}\boldsymbol{\nu}_u, \boldsymbol{\nu}_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ and the Lebesgue measure. For that, we need to modify the definition of $\mathcal{M}$.

2.2 Heuristic

By an heuristic way, we show the relation between $\mathcal{S}_{\mathbf{M}}(S)$ and $\mathcal{M}_{\mathbf{M}}(S)$. In the proof of Theorem 1.1, the author reduces the comparison between $\mathcal{H}^{n-1}(S)$ and $\mathcal{M}(S)$ to the case where S is a $(n-1)$-dimensional simplex and then, with approximation arguments, he deduces the general case. Following the same spirit, we may compute $\mathcal{S}_{\mathbf{M}}(S)$, with S a simplex, to deduce an adapted definition of the Minkowski content.

Thus, in this section, we assume that $\mathbf{M}$ does not depend on $x \in \Omega$ and $S \subset \mathbb{R}^n$ is a $(n-1)$-dimensional simplex generated by the vector $\{\mathbf{v_1}, \ldots, \mathbf{v}_{n-1}\}$. We denote by $\boldsymbol{\nu}$ an unitary and normal vector to S. Then, we have

$$\int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} = \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathcal{H}^{n-1}(S) = \frac{\mathcal{L}^n(K)}{\langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}}}, \tag{2.2}$$

where K is the n-dimensional simplex K generated by $\{\mathbf{v_1}, \ldots, \mathbf{v}_{n-1}, \mathbf{M}\boldsymbol{\nu}\}$ (see Figure 2.2).

Considering the obvious relations

$$\forall i \in \{1, \ldots, n-1\}, \quad \langle \mathbf{M}^{-1}(\mathbf{M}\boldsymbol{\nu}), \mathbf{v}_i\rangle = 0, \quad \langle \mathbf{M}^{-1}(\mathbf{M}\boldsymbol{\nu}), \mathbf{M}\boldsymbol{\nu}\rangle = \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle,$$

we observe that K is the *thickening* of S according to the metric associated to the scalar product $(\mathbf{v}, \mathbf{w}) \to \langle \mathbf{M}^{-1}\mathbf{v}, \mathbf{w}\rangle$ with a corresponding thickness equal to $\langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}}$.That the reason why we introduce the following metric

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}}$$

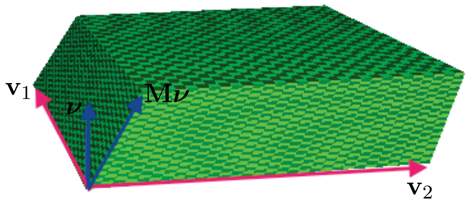

Figure 2.2: Simplex K

and its associated integrated distance

$$\forall (x,y)\in\Omega^2, \quad \mathrm{dist}_\phi(x,y) = \inf\left\{\int_0^1 \phi(\gamma,\dot\gamma)\mathrm{d}t\colon \begin{array}{l}\gamma\in W^{1,1}([0;1];\Omega),\\ \gamma(0)=x, \gamma(1)=y\end{array}\right\}.$$

Neglecting the *thickening* at the boundary of S (yellow part of Figure 2.3), ratio (2.2) may be interpreted as the following

$$\int_S \langle \mathbf{M}\boldsymbol{\nu},\boldsymbol{\nu}\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1} \approx \frac{\mathcal{L}^n(\{x\colon \mathrm{dist}_\phi(x,S) < \langle \mathbf{M}\boldsymbol{\nu},\boldsymbol{\nu}\rangle^{\frac{1}{2}}\})}{2\langle \mathbf{M}\boldsymbol{\nu},\boldsymbol{\nu}\rangle^{\frac{1}{2}}}.$$

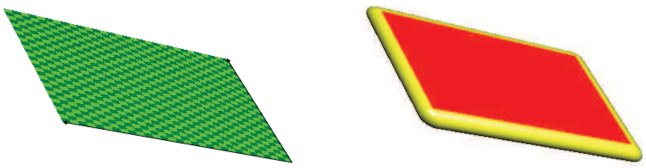

Figure 2.3: Simplex S and $\{x\colon \mathrm{dist}_\phi(x,S) < \langle \mathbf{M}\boldsymbol{\nu},\boldsymbol{\nu}\rangle^{\frac{1}{2}}\}$

Thus, the associated anisotropic Minkowski $(n-1)$-dimensional upper and lower content may be defined as the limits

$$\begin{aligned}\mathcal{M}^\star_{\mathbf{M}}(S) &= \limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x\in\Omega\colon \mathrm{dist}_\phi(x,S)<\rho\})}{2\rho},\\ \mathcal{M}_{\star\mathbf{M}}(S) &= \liminf_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x\in\Omega\colon \mathrm{dist}_\phi(x,S)<\rho\})}{2\rho}.\end{aligned}$$

2.3 Geometric Measure Theory Framework

The geometric measure theory framework is mainly extracted from [1].

Definition 2.1 (Federer, 3.2.1)**.** *Let f maps a subset of $\mathbb{R}^{n-1}$ onto $\mathbb{R}^n$, the $(n-1)$-dimensional Jacobian is defined by*

$$\boldsymbol{J}_{n-1}(f)(a) = \left|\bigwedge_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(a)\right|$$

whenever f is differentiable at a.

Theorem 2.1 (Area Formula, Federer, 3.2.3)**.** *Suppose $f:\mathbb{R}^{n-1}\to\mathbb{R}^n$ is Lipshitzian. If u is an $\mathcal{L}^{n-1}$-integrable function, then*

$$\int_{\mathbb{R}^{n-1}} u(x)\boldsymbol{J}_{n-1}(f)(x)d\mathcal{L}^{n-1}(x) = \int_{\mathbb{R}^n}\sum_{x\in f^{-1}\{y\}} u(x)d\mathcal{H}^{n-1}(y).$$

We are interested in the class of subset of $\mathbb{R}^n$ which are $(n-1)$-rectifiable.

Definition 2.2 (Federer, 3.2.14)**.** *The set $E \subset \mathbb{R}^n$ is $(n-1)$-rectifiable if there exists a Lipshitzian function $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ mapping some bounded subset of $\mathbb{R}^n$ onto E.*

The following result is useful to get an univalent parametrization of a rectifiable set.

Corollary 2.1 (Federer, 3.2.4)**.** *For every $\mathcal{L}^{n-1}$ measurable set A, there exists a Borel set*

$$B \subset A \cap \{x \colon \boldsymbol{J}_{n-1}(f)(x) > 0\}$$

such that $f|_B$ is univalent and $\mathcal{H}^{n-1}(f(A) \setminus f(B)) = 0$.

The last result is extracted from [2] and is specific to the computation of the anisotropic n-dimensional Hausdorff measure.

Theorem 2.2 (Bellettini, 4.1)**.** *If $\phi : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ be a continuous function which satisfies*

i) $\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \forall t \in \mathbb{R}, \quad \phi(x, t\mathbf{v}) = |t|\phi(x, \mathbf{v}),$

ii) $\exists \lambda > 0, \exists \Lambda > 0, \forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \lambda|\mathbf{v}| \le \phi(x, \mathbf{v}) \le \Lambda|\mathbf{v}|.$

Let $\mathcal{H}^n_\phi$ be the n-dimensional Hausdorff measure associated to the metric introduced in (1.2). Then, for all Borel set $E \subset \mathbb{R}^n$, we have

$$\mathcal{H}^n_\phi(E) = \int_E \frac{\mathcal{H}^n(B(0,1))}{\mathcal{H}^n(B_\phi(x,1))}\, dx,$$

where $B_\phi(x, 1)$ is the unit ball centered at x for the metric ϕ.

3 Proof of Theorem 1.2

3.1 Homogeneous Case

In this section, as for the heuristic, we assume that the metric is homogeneous, i.e. $\mathbf{M}$ does not depend on $x \in \Omega$. We denote by $\mathbf{M}_0$ its common value and by ϕ the following norm

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \phi(\mathbf{v}) = \langle \mathbf{M}_0^{-1}\mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}}. \tag{3.1}$$

We denote by $\mathcal{H}^n_\phi$ the n-dimensional Hausdorff measure associated to this norm. We prove the following.

Theorem 3.1. *If S is a closed and $(n-1)$-rectifiable subset of $\mathbb{R}^n$ and $\mathbf{M}_0 \in S_n^+(\mathbb{R})$. Then we have*

$$\mathcal{M}_{\mathbf{M}_0}(S) = \int_S \langle \mathbf{M}_0 \boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}}\, d\mathcal{H}^{n-1},$$

where $\boldsymbol{\nu}$ is an unitary and normal vector to S.

We decompose the proof in three Lemmas. As $\mathbf{M}_0$ is symmetric positive definite, $\mathbb{R}^n$ may be viewed as an euclidean space according to the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$. So, Theorem 1.1 may be directly applied. The three following Lemmas consist in the change of variable from $\mathbb{R}^n$ endowed with the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$ to $\mathbb{R}^n$ endowed with the canonical scalar product.

Lemma 3.1. *For $\mathbf{M}_0 \in S_n^+(\mathbb{R})$ and ϕ defined by (3.1), we have*

$$\forall E \in \mathcal{B}(\mathbb{R}^n), \quad \mathcal{L}^n(E) = \sqrt{det(\mathbf{M}_0)}\mathcal{H}^n_\phi(E).$$

Proof. As $\mathbf{M}_0$ is a symmetric positive definite matrix, there exists $P \in \mathrm{O}_n(\mathbb{R})$ and $(\lambda_i)_{i=1\ldots n}$ such that $\lambda_i > 0$ for any $i \in \{1, \ldots, n\}$ and

$$\mathbf{M}_0 = P \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{pmatrix} P^{-1}. \tag{3.2}$$

We denote by $(c_i)_{i=1\ldots n}$ the canonical basis of $\mathbb{R}^n$, then the family $(\sqrt{\lambda_i}Pc_i)_{i=1\ldots n}$ is a orthonormal basis for the scalar product $\langle \mathbf{M}_0^{-1}\cdot, \cdot\rangle$. So, the linear application $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ characterized by

$\varphi(c_i) = \sqrt{\lambda_i} P c_i$ for any $i \in \{1, \ldots, n\}$, is an isomorphism which satisfies $\varphi(B(0,1)) = B_\phi(0,1)$ and $\det(\varphi) = \sqrt{\det(\mathbf{M}_0)}$. It gives

$$\begin{aligned}\mathcal{H}^n(B_\phi(0,1)) &= \int_{B(0,1)} |\det(\varphi)| \mathrm{d}x, \\ &= \sqrt{\det(\mathbf{M}_0)} \mathcal{L}^n(B(0,1))\end{aligned}$$

As ϕ satisfies all the conditions of Theorem 2.2, it gives

$$\forall E \in \mathcal{B}(\mathbb{R}^n), \quad \mathcal{H}^n_\phi(E) = \frac{\mathcal{H}^n(B(0,1))}{\sqrt{\det(\mathbf{M}_0)} \mathcal{L}^n(B(0,1))} \mathcal{L}^n(E).$$

As $\mathcal{H}^n = \mathcal{L}^n$, it concludes the proof of Lemma 3.1. □

Denoting $e_i = \sqrt{\lambda_i} P c_i$, then $(e_i)_{i=1,\ldots,n}$ is an orthonormal basis of $\mathbb{R}^n$ for the scalar product $\langle \mathbf{M}_0^{-1} \cdot, \cdot \rangle$. We may define the associated vectorial product of $(\mathbf{v}_i)_{i=1,\ldots,n-1}$ as the vector $\bigwedge_{\phi,i=1}^{n-1} \mathbf{v}_i$ characterized by:

$$\forall \mathbf{w} \in \mathbb{R}^n, \quad \begin{vmatrix} \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_1 \rangle & \ldots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_1 \rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_1 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_n \rangle & \ldots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_n \rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_n \rangle \end{vmatrix} = \left\langle \mathbf{M}_0^{-1} \left(\wedge_{\phi,i=1}^{n-1} \mathbf{v}_i \right), \mathbf{w} \right\rangle. \quad (3.3)$$

Lemma 3.2. *If $\mathbf{M}_0 \in S_n^+(\mathbb{R})$ and ϕ defined by (3.1), then we have*

$$\wedge_{\phi,i=1}^{n-1} \mathbf{v}_i = \frac{\mathbf{M}_0(\wedge_{i=1}^{n-1} \mathbf{v}_i)}{\sqrt{det(\mathbf{M}_0)}}.$$

Proof. As $P \in \mathrm{O}_n(\mathbb{R})$, then $(Pc_i)_{i=1}^n$ is an orthonormal basis for the usual scalar product and, for any $\mathbf{w} \in \mathbb{R}^n$, we have

$$\begin{vmatrix} \langle \mathbf{v}_1, Pc_1 \rangle & \ldots & \langle \mathbf{v}_{n-1}, Pc_1 \rangle & \langle \mathbf{w}, Pc_1 \rangle \\ \vdots & \vdots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots \\ \langle \mathbf{v}_1, Pc_n \rangle & \ldots & \langle \mathbf{v}_{n-1}, Pc_n \rangle & \langle \mathbf{w}, Pc_n \rangle \end{vmatrix} = \left\langle \wedge_{i=1}^{n-1} \mathbf{v}_i, \mathbf{w} \right\rangle.$$

According to (3.2), we have

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{v}, Pc_i \rangle = \sqrt{\lambda_i} \langle \mathbf{M}_0^{-1}\mathbf{v}, \sqrt{\lambda_i} P c_i \rangle$$

and then

$$\sqrt{\lambda_1 \ldots \lambda_n} \begin{vmatrix} \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_1 \rangle & \ldots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_1 \rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_1 \rangle \\ \vdots & & \vdots & \vdots \\ \vdots & & \vdots & \vdots \\ \langle \mathbf{M}_0^{-1}\mathbf{v}_1, e_n \rangle & \ldots & \langle \mathbf{M}_0^{-1}\mathbf{v}_{n-1}, e_n \rangle & \langle \mathbf{M}_0^{-1}\mathbf{w}, e_n \rangle \end{vmatrix} = \left\langle \mathbf{M}_0^{-1} \left(\mathbf{M}_0 \wedge_{i=1}^{n-1} \mathbf{v}_i \right), \mathbf{w} \right\rangle.$$

According to (3.3), it concludes the proof of Lemma 3.2.

□

As for the vectorial product, we may define an anisotropic $(n-1)$-dimensional Jacobian by

$$\mathbf{J}_{n-1}^\phi(f)(a) = \left| \bigwedge_{\phi,i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right|$$

whenever f is differentiable at a. The following result is a straightforward consequence of Lemma 3.2.

Lemma 3.3. *If $\mathbf{M}_0 \in S_n^+(\mathbb{R})$ and ϕ defined by (3.1), then we have*

$$\boldsymbol{J}_{n-1}^{\phi}(f)(a) = \frac{1}{\sqrt{det(\mathbf{M}_0)}} \left\langle \mathbf{M}_0 \left(\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right), \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(a) \right\rangle^{\frac{1}{2}}.$$

Now we give the proof of Theorem 3.1.

Proof. Let S be a subset $(n-1)$-rectifiable and closed. As φ, introduced in the proof of Lemma 3.1, is an automorphism, then S is also $(n-1)$-rectifiable and closed for $\mathbb{R}^n$ endowed with the scalar product $\langle \mathbf{M}_0^{-1} \cdot, \cdot \rangle$. According to Lemma 3.1, we have

$$\frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_{\phi}(x, S) < \rho\})}{2\rho} = \sqrt{\det(\mathbf{M}_0)} \frac{\mathcal{H}_{\phi}^n(\{x \in \Omega \colon \mathrm{dist}_{\phi}(x, S) < \rho\})}{2\rho}. \tag{3.4}$$

We may apply Theorem 1.1 in the euclidean space $(\mathbb{R}^n, \langle \mathbf{M}_0^{-1} \cdot, \cdot \rangle)$, it gives

$$\lim_{\rho \to 0^+} \frac{\mathcal{H}_{\phi}^n(\{x \in \Omega \colon \mathrm{dist}_{\phi}(x, S) < \rho\})}{2\rho} = \mathcal{H}_{\phi}^{n-1}(S). \tag{3.5}$$

As S is rectifiable, there exists a Lipschitzian function $f : \mathbb{R}^{n-1} \to \mathbb{R}^n$ and a bounded subset $A \subset \mathbb{R}^{n-1}$ such that $S \subset f(A)$. According to Corollary 2.1, there exists a Borel set

$$B \subset A \cap \{x \colon \mathbf{J}_{n-1}^{\phi}(f)(x) > 0\}$$

such that $f_{|B}$ is univalent and $\mathcal{H}^{n-1}(f(A) \setminus f(B)) = 0$. We denote by $C = f^{-1}(S) \cap B$ and then $\mathcal{H}_{\phi}^{n-1}(f(C)) = \mathcal{H}_{\phi}^{n-1}(S)$. As $f_{|C}$ is univalent, according to Area formula 2.1 with $u = \mathbf{1}_C$, we have

$$\mathcal{H}_{\phi}^{n-1}(S) = \int_C \mathbf{J}_{n-1}^{\phi}(f)(x) \mathrm{d}x. \tag{3.6}$$

For any $x \in C$ we have $\mathbf{J}_{n-1}^{\phi}(f)(x) > 0$ and then $\mathbf{J}_{n-1}(f)(x) > 0$. According to Lemma 3.3, we may write

$$\begin{aligned}
\mathbf{J}_{n-1}^{\phi}(f)(x) &= \frac{\mathbf{J}_{n-1}^{\phi}(f)(x)}{\mathbf{J}_{n-1}(f)(x)} \mathbf{J}_{n-1}(f)(x), \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}} \left\langle \mathbf{M}_0 \left(\frac{\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x)}{\left| \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x) \right|} \right), \frac{\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x)}{\left| \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x) \right|} \right\rangle^{\frac{1}{2}} \mathbf{J}_{n-1}(f)(x).
\end{aligned}$$

We denote

$$u(x) = \left\langle \mathbf{M}_0 \left(\frac{\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x)}{\left| \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x) \right|} \right), \frac{\bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x)}{\left| \bigwedge_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x) \right|} \right\rangle^{\frac{1}{2}}.$$

According to Lemma 3.2.25 in [1], we remark that $u(x) = \langle \mathbf{M}_0 \boldsymbol{\nu}(f(x)), \boldsymbol{\nu}(f(x)) \rangle^{\frac{1}{2}}$ where $\boldsymbol{\nu}(f(x))$ is an unitary and orthogonal vector to $f(C)$ at $f(x)$. Applying Area formula 2.1, this time in $\mathbb{R}^n$ endowed with its canonical euclidean structure gives

$$\begin{aligned}
\int_C \mathbf{J}_{n-1}^{\phi}(f)(x) \mathrm{d}x &= \frac{1}{\sqrt{\det(\mathbf{M}_0)}} \int_C u(x) \mathbf{J}_{n-1}(f)(x) \mathrm{d}x, \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}} \int_{f(C)} \langle \mathbf{M}_0 \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\
&= \frac{1}{\sqrt{\det(\mathbf{M}_0)}} \int_S \langle \mathbf{M}_0 \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.
\end{aligned}$$

According to (3.4), (3.5) and (3.6) we may conclude

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega \colon \mathrm{dist}_{\phi}(x, S) < \rho\})}{2\rho} = \int_S \langle \mathbf{M}_0 \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

□

3.2 Inhomogeneous setting

In this section, we remove the homogeneity assumption and using a piecewise constant approximation of $\mathbf{M}$ we can prove the following.

Theorem 3.2. *If $\Omega \subset \mathbb{R}^n$ is bounded, S is a closed $(n-1)$-rectifiable subset of Ω and $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ satisfies Hölder-regularity condition:*

$$(H): \qquad \exists \alpha > 0, \exists \theta > 0, \forall (x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \leq \theta |x-y|^\alpha,$$

then we have

$$\mathcal{M}_{\mathbf{M}}(S) = \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1},$$

where $\boldsymbol{\nu}$ is an unitary, normal vector to S.

The proof is divided in three propositions.

Proposition 3.1. *Let Ω and $\mathbf{M}$ be as in Theorem 3.1. Then $\mathbf{M}$ satisfies the following Ellipticity condition:*

$$(E): \qquad \exists \lambda > 0, \exists \Lambda > 0, \forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \lambda |\mathbf{v}|^2 \leq \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \leq \Lambda |\mathbf{v}|^2.$$

Proof. As $\mathbf{M}(x)$ is symmetric positive definite for any $x \in \Omega$, we have

$$\forall x \in \Omega, \exists \lambda(x) > 0, \exists \Lambda(x) > 0, \forall \mathbf{v} \in \mathbb{R}^n, \quad \lambda(x)|\mathbf{v}|^2 \leq \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \leq \Lambda(x)|\mathbf{v}|^2.$$

According to (H), $\mathbf{M}$ is uniformly continuous, we may extend it in a continuous way to $\overline{\Omega}$. As Ω is bounded, then $\overline{\Omega}$ is compact and the previous inequalities remain true with (λ, Λ) independent of $x \in \Omega$. □

Proposition 3.2. *Let Ω, S and $\mathbf{M}$ be as in Theorem 3.1. Then, we have*

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1}.$$

Proof. We may assume that $\int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. According to Proposition 3.1, there exists $\lambda > 0$ such that

$$\lambda^{\frac{1}{2}} \mathcal{H}^{n-1}(S) \leq \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

and then $\mathcal{H}^{n-1}(S)$ is also finite. For $t \in \mathbb{R}$ and $i \in \{1, \ldots, n\}$, we denote $\Pi_t^i = \{x \in \Omega \colon \langle x, e_i\rangle = t\}$. Thus, for $k \in \mathbb{N}$ fixed, $\left\{t \in \mathbb{R} \colon \mathcal{H}^{n-1}(S \cap \Pi_t^i) > \frac{1}{k}\right\}$ is finite and then, for any $i \in \{1, \ldots, n\}$, the set $\left\{t \in \mathbb{R} \colon \mathcal{H}^{n-1}(S \cap \Pi_t^i) > 0\right\}$ is at most countable. So, for $\eta > 0$ fixed, there exists a covering $\mathcal{K}$ of Ω by cubes with diameter less than η and with disjoint interiors, such that

$$\forall K \in \mathcal{K}, \quad \mathcal{H}^{n-1}(S \cap \partial K) = 0. \tag{3.7}$$

For any $K \in \mathcal{K}$, we choose $a_K \in K$ and we set

$$\begin{cases} \forall K \in \mathcal{K}, \forall x \in K, & \widetilde{\mathbf{M}}(x) = \mathbf{M}(a_K), \\ \forall (x, \mathbf{v}) \in K \times \mathbb{R}^n, & \widetilde{\phi}(x, \mathbf{v}) = \langle (\mathbf{M}(a_K))^{-1}\mathbf{v}, \mathbf{v}\rangle. \end{cases} \tag{3.8}$$

For $K \in \mathcal{K}$ and $r > 0$, we set $K_r = \{x \colon \mathrm{dist}(x, \partial K) \leq r\}$. We have the following decomposition

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \sum_{K \in \mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S \cap K_r) + \sum_{K \in \mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S \cap (K \setminus K_r)). \tag{3.9}$$

In *Claim 1* and *Claim 2*, we determine an upper bound for the two previous sums.

Claim 1: We have

$$\sum_{K \in \mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S \cap K_r) \leq \lambda^{-\frac{n}{2}} \Lambda^{\frac{1}{2}} \sum_{K \in \mathcal{K}} \mathcal{H}^{n-1}(S \cap K_r).$$

According to Proposition 3.1, there exists $\lambda > 0$ and $\Lambda > 0$ such that

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \Lambda^{-\frac{1}{2}} |\mathbf{v}| \leq \langle \mathbf{M}(x)^{-1} \mathbf{v}, \mathbf{v} \rangle^{\frac{1}{2}} \leq \lambda^{-\frac{1}{2}} |\mathbf{v}|. \tag{3.10}$$

Let $(x, y) \in \Omega^2$ and $\gamma \in \mathrm{W}^{1,1}([0;1]; \Omega)$ such that $\gamma(0) = x$ and $\gamma(1) = y$, then we have

$$\Lambda^{-\frac{1}{2}} \int_0^1 |\dot{\gamma}| \mathrm{d}t \leq \int_0^1 \phi(\gamma, \dot{\gamma}) \mathrm{d}t \leq \lambda^{-\frac{1}{2}} \int_0^1 |\dot{\gamma}| \mathrm{d}t.$$

According to (1.2) and (3.10), it gives

$$\forall (x, y) \in \Omega^2, \quad \Lambda^{-\frac{1}{2}} \mathrm{dist}(x, y) \leq \mathrm{dist}_\phi(x, y) \leq \lambda^{-\frac{1}{2}} \mathrm{dist}(x, y) \tag{3.11}$$

and then

$$\forall x \in \Omega, \quad B(x, \lambda^{\frac{1}{2}}) \subset B_\phi(x, 1) \subset B(x, \Lambda^{\frac{1}{2}}).$$

As $\mathcal{H}^n(B(x,t)) = t^n \mathcal{H}^n(B(0,1))$, we have

$$\forall x \in \Omega, \quad \Lambda^{-\frac{n}{2}} \leq \frac{\mathcal{H}^n(B(0,1))}{\mathcal{H}^n(B_\phi(x,1))} \leq \lambda^{-\frac{n}{2}}$$

and with Theorem 2.2, it yields

$$\forall E \in \mathcal{B}(\Omega), \quad \Lambda^{-\frac{n}{2}} \mathcal{H}^n(E) \leq \mathcal{H}^n_\phi(E) \leq \lambda^{-\frac{n}{2}} \mathcal{H}^n(E). \tag{3.12}$$

Moreover, (3.11) implies $\{x \colon \mathrm{dist}_\phi(x, S \cap K_r) < \rho\} \subset \{x \colon \mathrm{dist}(x, S \cap K_r) < \Lambda^{\frac{1}{2}} \rho\}$ and then we have

$$\begin{aligned}
\mathcal{M}^\star_{\mathbf{M}}(S \cap K_r) &= \limsup_{\rho \to 0^+} \frac{\mathcal{H}^n_\phi(\{x \colon \mathrm{dist}_\phi(x, S \cap K_r) < \rho\})}{2\rho}, \\
&\leq \lambda^{-\frac{n}{2}} \limsup_{\rho \to 0^+} \frac{\mathcal{H}^n(\{x \colon \mathrm{dist}(x, S \cap K_r) < \Lambda^{\frac{1}{2}} \rho\})}{2\rho}, \\
&\leq \lambda^{-\frac{n}{2}} \Lambda^{\frac{1}{2}} \mathcal{M}^\star(S \cap K_r).
\end{aligned}$$

As S is closed and $(n-1)$-rectifiable, so is $S \cap K_r$. We may apply Theorem 1.1 to get $\mathcal{M}^\star(S \cap K_r) = \mathcal{H}^{n-1}(S \cap K_r)$ and *Claim 1* is proved.

Claim 2: We have

$$\sum_{K \in \mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S \cap (K \setminus K_r)) \leq (1 + \theta' \eta^\alpha) \int_S \langle \mathbf{M} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1},$$

where θ' is a constant which depends only on $\mathbf{M}$ uniformly with respect to $x \in \Omega$ and $\alpha > 0$ is given by Holder regularity assumption (H).

As the set of symmetric matrices that satisfy ellipticity condition (E) is a compact subset of $\mathrm{GL}_n(\mathbb{R})$ and the inversion is a continuous application on $\mathrm{GL}_n(\mathbb{R})$, there exists a constant $m > 0$ which depends only on (λ, Λ) such that

$$\mathbf{M}_1, \mathbf{M}_2 \text{ satisfy } (H) \Rightarrow \|\mathbf{M}_1^{-1} - \mathbf{M}_2^{-1}\| \leq m \|\mathbf{M}_1 - \mathbf{M}_2\|.$$

Let $(x, y) \in \Omega^2$ and $\gamma \in \mathrm{W}^{1,1}([0;1]; \Omega)$ such that $\gamma(0) = x$ and $\gamma(1) = y$, according to Holder regularity assumption (H) we estimate

$$\begin{aligned}
\left| \int_0^1 \langle \mathbf{M}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} \mathrm{d}t - \int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} \mathrm{d}t \right| &\leq \int_0^1 \frac{|\langle (\mathbf{M}^{-1}(\gamma) - \widetilde{\mathbf{M}}^{-1}(\gamma)) \dot{\gamma}, \dot{\gamma} \rangle|}{\langle \mathbf{M}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} + \langle \mathbf{M}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}}} \mathrm{d}t, \\
&\leq \int_0^1 \frac{\|\mathbf{M}^{-1}(\gamma) - \widetilde{\mathbf{M}}^{-1}(\gamma))\| \langle \dot{\gamma}, \dot{\gamma} \rangle}{\langle \mathbf{M}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} + \langle \mathbf{M}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}}} \mathrm{d}t, \\
&\leq \frac{m\theta}{2\Lambda^{-1}} \eta^\alpha \int_0^1 \langle \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} \mathrm{d}t, \\
&\leq \frac{m\theta}{2\Lambda^{-\frac{3}{2}}} \eta^\alpha \int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma) \dot{\gamma}, \dot{\gamma} \rangle^{\frac{1}{2}} \mathrm{d}t.
\end{aligned}$$

We denote $\theta' = \frac{m\theta}{2\Lambda^{-\frac{3}{2}}}$, this gives

$$\left|\int_0^1 \langle \mathbf{M}^{-1}(\gamma)\dot{\gamma}, \dot{\gamma}\rangle^{\frac{1}{2}} \mathrm{d}t\right| \leq (1+\theta'\eta^\alpha) \int_0^1 \langle \widetilde{\mathbf{M}}^{-1}(\gamma)\dot{\gamma}, \dot{\gamma}\rangle^{\frac{1}{2}} \mathrm{d}t,$$

so that

$$\{x\colon \mathrm{dist}_\phi(x, S\cap(K\setminus K_r)) < \rho\} \subset \{x\colon \mathrm{dist}_{\widetilde{\phi}}(x, S\cap(K\setminus K_r)) < (1+\theta'\eta^\alpha)\rho\}, \tag{3.13}$$

and then

$$\mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)) \leq (1+\theta'\eta^\alpha)\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)). \tag{3.14}$$

As $\widetilde{\phi}$ is an homogeneous metric in a neighborhood of $K\setminus K_r$ as in Section 3.1, we may apply Theorem 3.1 to obtain

$$\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) = \int_{S\cap\overline{(K\setminus K_r)}} \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

and

$$\mathcal{M}^\star_{\widetilde{\mathbf{M}}}(S\cap(K\setminus K_r)) \leq \int_{S\cap K} \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

According to (3.14), we get

$$\sum_{K\in\mathcal{K}} \mathcal{M}^\star_{\mathbf{M}}(S\cap(K\setminus K_r)) \leq (1+\theta'\eta^\alpha) \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1},$$

so it concludes the proof of *Claim 2*.

Applying *Claim 1* and *Claim 2* to the decomposition (3.9), we have

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \lambda^{-\frac{n}{2}}\Lambda^{\frac{1}{2}} \sum_{K\in\mathcal{K}} \mathcal{H}^{n-1}(S\cap K_r) + (1+\theta'\eta^\alpha) \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

Taking the limit $r\to 0^+$ gives

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \lambda^{-\frac{n}{2}}\Lambda^{\frac{1}{2}} \sum_{K\in\mathcal{K}} \mathcal{H}^{n-1}(S\cap \partial K) + (1+\theta'\eta^\alpha) \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

Applying (3.7), and taking the limit as $\eta\to 0^+$ concludes the proof of Proposition 3.2

$$\mathcal{M}^\star_{\mathbf{M}}(S) \leq \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

□

Proposition 3.3. *Let Ω, S and $\mathbf{M}$ as in Theorem 3.1. Then, we have*

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq \int_S \langle \mathbf{M}\boldsymbol{\nu}, \boldsymbol{\nu}\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

Proof. Let $\eta > 0$ and $\mathcal{K}$ be the same partition of Ω as in (3.7) and (3.8). For $r>0$ and $K\in\mathcal{K}$, we still denote $K_r = \{x\colon \mathrm{dist}(x,\partial K)\leq r\}$. As $\mathcal{K}$ is finite and

$$(K,L)\in\mathcal{K}^2, K\neq L \quad\Rightarrow\quad \mathrm{dist}(K_r, L_r) > 0,$$

then we have

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq \sum_{K\in\mathcal{K}} \mathcal{M}_{\star\mathbf{M}}(S\cap(K\setminus K_r)) \tag{3.15}$$

With the same proof as for (3.13), we have

$$\{x\colon \mathrm{dist}_{\widetilde{\phi}}(x, S\cap(K\setminus K_r)) < \rho\} \subset \{x\colon \mathrm{dist}_\phi(x, S\cap(K\setminus K_r)) < (1+\theta'\eta^\alpha)\rho\}$$

and then

$$\frac{\mathcal{L}^n(\{x\in\Omega\colon \mathrm{dist}_{\widetilde{\phi}}(x, S\cap(K\setminus K_r)) < \rho\})}{2\rho} \leq \frac{\mathcal{L}^n(\{x\in\Omega\colon \mathrm{dist}_\phi(x, S\cap(K\setminus K_r)) < (1+\theta'\eta^\alpha)\rho\})}{2\rho}.$$

Passing to the lim inf with $\rho \to 0^+$ gives

$$\mathcal{M}_{\star\widetilde{\mathbf{M}}}(S \cap (K \setminus K_r)) \leq (1 + \theta' \eta^\alpha) \mathcal{M}_{\star\mathbf{M}}(S \cap (K \setminus K_r)).$$

As $\widetilde{\mathbf{M}}$ is constant in a neighborhood of $K \setminus K_r$ as in Section 3.1, we may apply Theorem 3.1 to obtain

$$\mathcal{M}_{\star\widetilde{\mathbf{M}}}(S \cap (K \setminus K_r)) = \int_{S \cap \overline{(K \setminus K_r)}} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

According to (3.15), we get

$$\begin{aligned} \mathcal{M}_{\star\mathbf{M}}(S) &\geq (1 + \theta' \eta^\alpha)^{-1} \sum_{K \in \mathcal{K}} \int_{S \cap \overline{(K \setminus K_r)}} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\ &\geq (1 + \theta' \eta^\alpha)^{-1} \int_{S \cap C_r} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \end{aligned}$$

where we have set $C_r = \bigcup_{K \in \mathcal{K}} \overline{(K \setminus K_r)}$. Remark that

$$r_1 < r_2 \Rightarrow C_{r_2} \subset C_{r_1}, \quad \bigcup_{r > 0} C_r = \Omega \setminus \left(\bigcup_{K \in \mathcal{K}} \partial K \right)$$

so, we have

$$\lim_{r \to 0^+} \int_{S \cap C_r} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} = \int_{S \setminus \cup \partial K} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

As $\mathcal{H}^{n-1}(\partial K) = 0$ for any $K \in \mathcal{K}$ (3.7), then

$$\lim_{r \to 0^+} \int_{S \cap C_r} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} = \int_{S} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

We deduce that

$$\mathcal{M}_{\star\mathbf{M}}(S) \geq (1 + \theta' \eta^\alpha)^{-1} \int_{S} \langle \widetilde{\mathbf{M}} \boldsymbol{\nu}, \boldsymbol{\nu} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

and passing to the limit as $\eta \to 0^+$ in this inequality concludes the proof.

□

4 Application to almost quasi minimizers of a free boundary problem

4.1 Definition and main result

We are interested in regularity for local minimizers of free discontinuity problems. The functional framework is the theory of BV and SBV functions that may be found in [7] for BV and [8] for SBV. We give the definition of *almost-quasi minimizer of a free discontinuity problem* introduced in [9].

Definition 4.1. *For $\sigma \geq 1$, $\alpha > 0$ and $c_\alpha > 0$, we say that $w \in SBV(\Omega)$ is a $(\sigma, \alpha, c_\alpha)$-almost-quasi minimizer of a free discontinuity problem, if there exists $\alpha > 0$ and $c_\alpha \geq 0$ such that*

$$\begin{gathered} v \in SBV(\Omega), \quad x \in \Omega, \quad \overline{B(x,r)} \subset \Omega, \quad [w \neq v] \subset B(x,r) \qquad \Rightarrow \\ \int_{B(x,r)} |\nabla w|^2 dx + \mathcal{H}^{n-1}(J_w \cap \overline{B(x,r)}) \leq \int_{B(x,r)} |\nabla v|^2 dx + \sigma \mathcal{H}^{n-1}(J_v \cap \overline{B(x,r)}) + c_\alpha r^{n-1+\alpha}, \end{gathered} \tag{4.1}$$

where J_u and J_w are the jump sets of u and w.

In particular, it is easy to see that a minimizer of $E_{\mathbf{M}}$, introduced in (2.1), is an almost quasi minimizer. The main result we introduce in this section is the following.

Theorem 4.1. *Let $u \in SBV(\Omega)$ be an almost quasi-minimizer of a free discontinuity problem (4.1), then we have*

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M} \nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

4.2 Proof of Theorem 4.1

To prove Thorem 4.1, we need the two following regularity results for the jump set of almost quasi minimizers which are extracted from [9].

Theorem 4.2. *Let u be an almost-quasi minimizer of a free discontinuity problem, then*

$$\mathcal{H}^{n-1}\left(\overline{J_u} \setminus J_u\right) = 0.$$

Theorem 4.3. *There exist constants β, ρ_0 such that for every $(\sigma, \alpha, c_\alpha)$ almost quasi-minimizer u, for every $x \in J_u$ and for every $0 < \rho < \rho_0$ such that $B(x, \rho) \subset \Omega$, we have*

$$\mathcal{H}^{n-1}(J_u \cap B(x, \rho)) \geq \beta \rho^{d-1}.$$

First, we prove the following.

Lemma 4.1. *Let u be an almost quasi-minimizer of a free discontinuity problem (4.1), then for any compact set $K \subset \overline{J_u} \cap \Omega$, we have*

$$\mathcal{M}^{\star}_{\mathbf{M}}(K) \leq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \, d\mathcal{H}^{n-1}.$$

Proof. We may assume that $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. According to ellipticity condition 3.1, we have

$$\lambda^{\frac{1}{2}} \mathcal{H}^{n-1}(J_u) \leq \int_{J_u \cap K} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

so, $\mathcal{H}^{n-1}(J_u)$ is finite too.

According to [7], Section 5.9, the set J_u is rectifiable up to a $\mathcal{H}^{n-1}$-negligible set $\mathcal{N}$. More precisely, there exists a countable family $(K_i)_{i \in \mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that

$$J_u = \mathcal{N} \cup \left(\bigcup_{i \in \mathbb{N}} K_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$. Let $K \subset \overline{J_u} \cap \Omega$ be compact, we have the decomposition

$$K = \left[K \cap \left(\overline{J_u} \setminus J_u\right)\right] \cup \left[K \cap \bigcup_{i=1}^{q} K_i\right] \cup \left[K \cap \bigcup_{i=q+1}^{\infty} K_i\right] \cup \left[K \cap \mathcal{N}\right].$$

Theorem 4.2 gives $\mathcal{H}^{n-1}(K \cap \left(\overline{J_u} \setminus J_u\right)) = 0$. Let $\delta > 0$ be fixed. As $\mathcal{H}^{n-1}(J_u)$ is finite, there exists $q \in \mathbb{N}$ such that

$$\mathcal{H}^{n-1}\left(K \cap \bigcup_{i=q+1}^{\infty} K_i\right) \leq \delta. \tag{4.2}$$

In the sequel, we omit the dependance with δ for the sake of simplicity. We set $S = K \cap \bigcup_{i=1}^{q} K_i$ and for $A \subset \mathbb{R}^n$ we adopt the following notation

$$A_\rho := \{x \colon \mathrm{dist}_\phi(x, A) < \rho\}.$$

Let $\tau > 0$ and $\rho > 0$ be fixed, we decompose $K = (K \setminus S_{\tau\rho}) \cup S_{\tau\rho}$ and denote $E = K \setminus S_{\tau\rho}$. So, we have

$$K_\rho \subset E_\rho \cup S_{(1+\tau)\rho},$$

and then

$$\frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \frac{\mathcal{L}^n(E_\rho)}{2\rho} + \frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho}. \tag{4.3}$$

The rest of the proof consists in computing an upper bound, when $\rho \to 0^+$, for the two terms in the right hand side of (4.3). As S is a closed and rectifiable set, Theorem 1.2 gives

$$\lim_{\rho \to 0^+} \frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho} = (1+\tau) \int_S \langle \mathbf{M}\boldsymbol{\nu}_u, \boldsymbol{\nu}_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$$

and then

$$\lim_{\rho\to 0^+}\frac{\mathcal{L}^n(S_{(1+\tau)\rho})}{2\rho}=(1+\tau)\int_{J_u}\langle\mathbf{M}\boldsymbol{\nu}_u,\boldsymbol{\nu}_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}. \tag{4.4}$$

In the fourth following *Claims*, we prove that $\limsup\frac{\mathcal{L}^n(E_\rho)}{2\rho}$ converges to 0 when δ converges to 0. The main tool is the regularity result given by Theorem 4.3.

Claim 1: There exists $p\in\mathbb{N}$ and $x_1,\dots,x_p\in E\cap J_u$ such that $E\subset\bigcup_{i=1}^p\overline{B_\phi(x_i,\tau\rho)}$ and

$$(i,j)\in\{1,\dots,p\}^2, i\neq j\Rightarrow \mathit{dist}_\phi(x_i,x_j)\geq\tau\rho. \tag{4.5}$$

We construct $(x_i)_i$ by an iterative way and we show that the number of iterations is finite. If $E=\emptyset$, then the result is obvious. Otherwise, there exists $\tilde{x}_1\in E$. As $E\subset\overline{J_u}$, there exists $(y_k)_k\subset J_u$ converging to $\tilde{x}_1$. As $\mathrm{dist}_\phi(\tilde{x_1},S)>\tau\rho$, there exists $k_0\in\mathbb{N}$ such that $\mathrm{dist}_\phi(y_{k_0},S)>\tau\rho$ and we set $x_1=y_{k_0}$.

Let us assume that there exits $x_1,\dots,x_p\in E\cap J_u$ which satisfy (4.5). If $E\subset\bigcup_{i=1}^p\overline{B_\phi(x_i,\tau\rho)}$, then the iterative process stops. Otherwise, there exists $\tilde{x}_{p+1}\in E\setminus\bigcup_{i=1}^p\overline{B_\phi(x_i,\tau\rho)}$. As $E\subset\overline{J_u}$, there exists $(y_k)_k\subset J_u$ converging to $\tilde{x}_{p+1}$. As $\mathrm{dist}_\phi(\tilde{x}_{p+1},S)>\tau\rho$ and $\mathrm{dist}_\phi(\tilde{x}_{p+1},\tilde{x}_i)>\tau\rho$ for any $i\in\{1,\dots,p\}$, there exists $k_p\in\mathbb{N}$ such that $\mathrm{dist}_\phi(y_{k_p},S)>\tau\rho$ and $\mathrm{dist}_\phi(y_{k_p},\tilde{x}_i)>\tau\rho$. We set $x_p=y_{k_p}$.

If the iterative process does not finish, then there exists a sequence $(x_i)_{i\in\mathbb{N}}\subset\Omega$ which satisfies (4.5). According to Ellipticity condition, we have

$$(i,j)\in\{1,\dots,p\}^2, i\neq j\Rightarrow |x_i-x_j|\geq\frac{\tau\rho}{\sqrt{\Lambda}}.$$

As Ω is bounded there exists a converging subsequence which is a contradiction. So, the iterative process is finite.

Claim 2: There exists a constant $c=c(n,\lambda,\Lambda)$ which only depends on the dimension n and the ellipticity coefficients λ,Λ such that

$$\forall x\in\Omega,\quad \#\left\{i\in\{1,\dots,p\}\colon x\in B_\phi(x_i,\tau\rho)\right\}\leq c(n,\lambda,\Lambda).$$

We denote

$$c(n,\lambda,\Lambda)=\sup\left\{q\in\mathbb{N}\colon \exists y_1,\dots,y_q\in\mathbb{R}^n,\quad \forall i,|y_i|\leq\frac{1}{\sqrt{\lambda}},\quad i\neq j\Rightarrow|y_i-y_j|\geq\frac{1}{\sqrt{\Lambda}}\right\}.$$

We consider a finite partition of $B\left(0,\frac{1}{\sqrt{\lambda}}\right)$ of parallelepipeds whose diameter is less than $\frac{1}{\sqrt{\Lambda}}$. Then, $c(n,\lambda,\Lambda)$ is finite and less than the cardinality of such partition. For $x\in\Omega$, we set $y_i=\frac{x_i-x}{\tau\rho}$. According to ellipticity condition, we have

$$x\in B_\phi(x_i,\tau\rho)\Rightarrow y_i\in B\left(0,\frac{1}{\sqrt{\lambda}}\right),\quad i\neq j\Rightarrow|y_i-y_j|\geq\frac{1}{\sqrt{\Lambda}}.$$

and then $\#\left\{i\in\{1,\dots,p\}\colon x\in B_\phi(x_i,\tau\rho)\right\}\leq c(n,\lambda,\Lambda)$.

Claim 3: We still denote by c a generic constant depending on (n,λ,Λ). We have

$$p\beta\rho^{n-1}\leq\frac{c}{\tau^{n-1}}\mathcal{H}^{n-1}(J_u\cap E_{\tau\rho}).$$

According to *Claim 2*, we have

$$\sum_{i=1}^p\mathbf{1}_{J_u\cap B_\phi(x_i,\tau\rho)}\leq c\mathbf{1}_{J_u\cap E_{\tau\rho}}.$$

Integrating with respect to $\mathcal{H}^{n-1}$ gives

$$\sum_{i=1}^{p} \mathcal{H}^{n-1}(J_u \cap B_\phi(x_i, \tau\rho)) \leq c\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}).$$

Ellipticity condition gives $B(x_i, \lambda^{\frac{1}{2}}\tau\rho)) \subset B_\phi(x_i, \tau\rho))$ and then

$$\sum_{i=1}^{p} \mathcal{H}^{n-1}(J_u \cap B(x_i, \lambda^{\frac{1}{2}}\tau\rho)) \leq c\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}). \tag{4.6}$$

According to Theorem 4.3, there exists $\beta > 0$ such that

$$\forall i \in \{1, \ldots, p\}, \quad \mathcal{H}^{n-1}(J_u \cap B(x_i, \lambda^{\frac{1}{2}}\tau\rho)) \geq \beta\lambda^{\frac{n-1}{2}}\tau^{n-1}\rho^{n-1}. \tag{4.7}$$

Inequalities (4.6) and (4.7) conclude the proof of *Claim 3*.

Claim 4: We have

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(E_\rho)}{2\rho} \leq \frac{(1+\tau)^n c}{\tau^{n-1}}\delta$$

where δ is given by (4.2), β by Theorem 4.3 and ω_n is the volume of the unit ball of $\mathbb{R}^n$.

According to *Claim 1*, we have $E \subset \bigcup_{i=1}^{p} B_\phi(x_i, \tau\rho)$ and then $E_\rho \subset \bigcup_{i=1}^{p} B_\phi(x_i, (1+\tau)\rho)$. Ellipticity condition gives

$$B_\phi(x_i, (1+\tau)\rho) \subset B(x_i, \Lambda^{\frac{1}{2}}(1+\tau)\rho),$$

it yields

$$\begin{aligned} \mathcal{L}^n(E_\rho) &\leq p\mathcal{L}^n(B_\phi(x_i, (1+\tau)\rho)), \\ &\leq p\mathcal{L}^n(B(x_i, \Lambda^{\frac{1}{2}}(1+\tau)\rho)), \\ &\leq p\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n\rho^n. \end{aligned}$$

As $E = K \setminus S_{\tau\rho}$, then we deduce $E_{\tau\rho} \subset K_{\tau\rho} \setminus S$ and *Claim 3* gives

$$\begin{aligned} \mathcal{L}^n(E_\rho) &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{\beta\tau^{n-1}}\rho\mathcal{H}^{n-1}(J_u \cap E_{\tau\rho}), \\ &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{\beta\tau^{n-1}}\rho\mathcal{H}^{n-1}(J_u \cap (K_{\tau\rho} \setminus S)). \end{aligned}$$

According to (4.2), we deduce

$$\begin{aligned} \limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(E_\rho)}{2\rho} &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{2\beta\tau^{n-1}}\mathcal{H}^{n-1}(J_u \cap (K \setminus S)), \\ &\leq \frac{\Lambda^{\frac{n}{2}}(1+\tau)^n\omega_n c}{2\beta\tau^{n-1}}\delta. \end{aligned}$$

Conclusion of the proof.

According to (4.3), (4.4) and *Claim 4*, we have

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \frac{(1+\tau)^n c}{\tau^{n-1}}\delta + (1+\tau)\int_{J_u} \langle \mathbf{M}\boldsymbol{\nu}_u, \boldsymbol{\nu}_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

Letting $\delta, \tau \to 0^+$ successively gives

$$\limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(K_\rho)}{2\rho} \leq \int_{J_u} \langle \mathbf{M}\boldsymbol{\nu}_u, \boldsymbol{\nu}_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1},$$

which concludes the proof of Lemma 4.1.

□

Now, we give the proof of Theorem 4.1.

Proof. We divide the proof in two inequalities.

Claim 1: We have

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \le \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}}\, d\mathcal{H}^{n-1}.$$

We assume that $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ is finite, otherwise the result is ensured. We set $\Omega_r = \{x \in \Omega \colon \mathrm{dist}(x, \partial\Omega) > r\}$. So, the following set is at most countable

$$\Pi = \left\{ r > 0 \colon \int_{J_u \cap \partial\Omega_r} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} > 0 \right\}.$$

In particular, for any $\delta > 0$, there exists a sequence $(r_i)_i \subset \Pi$ strictly decreasing to 0^+ and such that

$$\int_{J_u \cap (\Omega \setminus \Omega_{r_0})} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \le \delta.$$

We consider the following partition

$$\Omega = \Omega_{r_0} \cup \left(\bigcup_{i=0}^{\infty} \Omega_{r_{i+1}} \setminus \Omega_{r_i} \right),$$

it gives

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \le \mathcal{M}^{\star}_{\mathbf{M}}(J_u \cap \Omega_{r_0}) + \sum_{i=0}^{\infty} \mathcal{M}^{\star}_{\mathbf{M}}(J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})).$$

As $\mathcal{M}^{\star}_{\mathbf{M}}(E) = \mathcal{M}^{\star}_{\mathbf{M}}(\overline{E})$ for any $E \subset \mathbb{R}^n$ such that $\overline{E} \subset \Omega$, we have

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \le \mathcal{M}^{\star}_{\mathbf{M}}(\overline{J_u \cap \Omega_{r_0}}) + \sum_{i=0}^{\infty} \mathcal{M}^{\star}_{\mathbf{M}}(\overline{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})}).$$

We may apply Lemma 4.1, with $K = \overline{J_u \cap \Omega_{r_0}}$ and $K = \overline{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})}$, it gives

$$\mathcal{M}^{\star}_{\mathbf{M}}(J_u) \le \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + \sum_{i=0}^{\infty} \int_{\overline{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})}} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

As $r_i \in \Pi$ for any $i \ge 1$, we have

$$\begin{aligned}
\mathcal{M}^{\star}_{\mathbf{M}}(J_u) &\le \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + \sum_{i=0}^{\infty} \int_{J_u \cap (\Omega_{r_{i+1}} \setminus \Omega_{r_i})} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\
&\le \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + \int_{J_u \cap (\Omega \setminus \Omega_{r_0})} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\
&\le \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} + \delta.
\end{aligned}$$

As $\delta > 0$ is arbitrary, it proves *Claim 1.*

Claim 2: We have

$$\mathcal{M}_{\star\mathbf{M}}(J_u) \ge \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}}\, d\mathcal{H}^{n-1}.$$

As for the proof of Lemma 4.1, there exists a countable family $(K_i)_{i\in\mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that

$$J_u = \mathcal{N} \cup \left(\bigcup_{i\in\mathbb{N}} K_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N})=0$. As $\bigcup_{i=0}^{q} K_i$ is rectifiable and closed, Theorem 1.2 gives

$$\mathcal{M}_{\star\mathbf{M}}\left(\bigcup_{i=0}^{q} K_i\right)=\int_{\bigcup_{i=0}^{q} K_i}\langle\mathbf{M}\nu_u,\nu_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}.$$

So, we deduce that

$$\mathcal{M}_{\star\mathbf{M}}\left(J_u\right)\geq\int_{\bigcup_{i=0}^{\infty} K_i}\langle\mathbf{M}\nu_u,\nu_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}$$

and then, as $\mathcal{H}^{n-1}(\mathcal{N})=0$, we conclude that

$$\mathcal{M}_{\star\mathbf{M}}(J_u)\geq\int_{J_u}\langle\mathbf{M}\nu_u,\nu_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}.$$

□

5 Conclusion

In this paper we have introduced the Minkowski content $\mathcal{M}_{\mathbf{M}}(S)$ adapted to the anisotropic measures of surface $\mathcal{S}_{\mathbf{M}}(S)$ and we have proved that under hypothesis of regularity on $\mathbf{M}$ (Hölder-Regularity) and regularity on S (closure and rectifiability) we get $\mathcal{M}_{\mathbf{M}}(S)=\mathcal{S}_{\mathbf{M}}(S)$. In particular, we apply this result to the case where $S=J_u$ is defined as the jump set of an almost-quasi-minimizer u. This result is the key tool to prove the upper Γ-limit of a Γ-convergence result. More precisely, in a forthcoming paper, for

$$E_{\mathbf{M},\varepsilon}(u,z)=\int_\Omega(u-g)^2\mathrm{d}x+\int_\Omega|\nabla u|^2(1-z)^2\mathrm{d}x+\int_\Omega\left(\varepsilon\langle\mathbf{M}\nabla z,\nabla z\rangle+\frac{z^2}{4\varepsilon}\right)\mathrm{d}x,$$

we will prove that $(E_{\mathbf{M},\varepsilon})_\varepsilon$ is Γ-convergent to

$$E_{\mathbf{M}}(u)=\int_\Omega(u-g)^2\mathrm{d}x+\int_\Omega|\nabla u|^2\mathrm{d}x+\int_{J_u}\langle\mathbf{M}\boldsymbol{\nu}_u,\boldsymbol{\nu}_u\rangle^{\frac{1}{2}}\mathrm{d}\mathcal{H}^{n-1}$$

when $\varepsilon\to 0^+$.

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Approximation of an Anisotropic Mumford-Shah Functional with Γ-convergence

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June 29, 2015

Introduction

In this paper, we provide an approximation of a new model we have introduced in [1] for the detection of thin structures in an image. We assume that the domain Ω is an open and bounded subset of $\mathbb{R}^n$, $n = 2$ or 3. At each point $x \in \Omega$ is associated an intensity $g(x)$ such that $g \in \mathrm{L}^\infty(\Omega)$. Let $\mathbf{M} : \Omega \to \mathrm{S}_n^+(\mathbb{R})$ be a field of symmetric positive definite matrices. We have introduced the following energy:

$$\mathcal{E}(u, K) = \int_{\Omega \setminus K} (u-g)^2 \mathrm{d}x + \int_{\Omega \setminus K} |\nabla u|^2 \mathrm{d}x + \int_K \langle \mathbf{M}\nu, \nu \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1},$$

where K is a compact hypersurface with class $\mathcal{C}^1$, $\nu : K \to \mathbb{S}^{n-1}$ an unitary and normal vector to K, and $u \in \mathrm{W}^{1,2}(\Omega \setminus K)$. For $\mathbf{M} \equiv \mathrm{Id}_n$, it corresponds to the so-called Mumford-Shah energy [2]. In this sense, $\mathcal{E}$ is a generalization in an *anisotropic* context, because $\mathcal{H}^{n-1}(K)$ is replaced by $\int_K \langle \mathbf{M}\nu, \nu \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ which depends on the orientation $\nu(x)$ of K at x. In order to show that the minimizing problem is well posed, we have introduced a relaxed formulation of this energy

$$E(u) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \tag{0.1}$$

where $u \in \mathrm{SBV}(\Omega)$ and J_u is its jump set. We have proved that the relaxed minimizing problem admits a solution and provides a minimizer for the initial energy $\mathcal{E}$.

In this paper, we are interested in the approximation of (0.1) by functionals for which the minimization process is more suitable for a numerical implementation. For $\varepsilon > 0$, we introduce

$$E_\varepsilon(u, z) = \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x.$$

As it has been done in [4] for the initial Mumford-Shah model, the function z takes its values in $[0; 1]$ and plays the role of control of the gradient of u.

In section 1, we introduce some conditions on Ω and $\mathbf{M}$, the functional framework and the main result of the paper. In section 2, is given the proof that the minimization of E_ε is a well posed problem. In section 3, we show that the family $(E_\varepsilon)_\varepsilon$ is an approximation of E when $\varepsilon \to 0^+$ in the sense of the Γ-convergence.

1 Definitions, tools and main result

We adopt the notations:

- $\langle \mathbf{v}_1, \mathbf{v}_2 \rangle \in \mathbb{R}$ for the canonical scalar product of $\mathbf{v}_1, \mathbf{v}_2 \in \mathbb{R}^n$,
- $|\mathbf{v}|$ for the euclidean norm of $\mathbf{v} \in \mathbb{R}^n$,
- $\|\mathbf{M}\|$ for the induced norm of $\mathbf{M} \in \mathrm{M}_n(\mathbb{R})$,

- $S_n^+(\mathbb{R}) \subset M_n(\mathbb{R})$ for the subset of symmetric definite positive matrices,
- $\mathbb{B}(\Omega)$ for the space of Borelian functions defined in Ω,
- $\mathcal{B}(\Omega)$ the class of Borelian subsets of Ω,
- $\mathcal{L}^k$ for the Lebesgue measure in $\mathbb{R}^k$,
- $\mathcal{H}^k$ for the k-dimensional Hausdorff measure,
- $\displaystyle ⨍_A f(x)\mathrm{dx} = \frac{1}{\mathcal{L}^n(A)}\int_A f(x)\mathrm{dx}$, for $A \in \mathcal{B}(\Omega)$ and $\mathcal{L}^n(A) > 0$.

1.1 The Domain Ω

We introduce a geometric constraint on the domain.

Definition 1.1. *We say that $\Omega \subset \mathbb{R}^n$ satisfies the **Reflexion condition** (R) if Ω is an open and bounded domain with Lipschitz regular boundary $\partial\Omega$ such that there exists a neighborhood U of $\partial\Omega$ and a bi-Lipschitzian homeomorphism $\varphi : U \cap \Omega \to U \setminus \overline{\Omega}$ such that*

$$\forall x \in \partial\Omega, \quad \lim_{y\to x} \varphi(y) = x.$$

If $\Omega =]-1;1[^n$, denoting $|\cdot|_\infty$ the ℓ^∞-norm, for $\delta > 0$, we set

$$U = \{x \in \mathbb{R}^n : |x|_\infty \in]1-\delta; 1+\delta[\}, \quad \forall x \in U,\ \varphi(x) = \frac{x}{|x|_\infty^2}.$$

Then Ω satisfies (R) and, by an affine composition, any parallelepiped of $\mathbb{R}^n$ satisfies (R). This condition is satisfied in the context of applications in *Image Processing* because the images are defined in a parallelepiped.

1.2 The associated metric ϕ

Definition 1.2. *We say that $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ satisfies **Hölder-regularity condition** (H) if*

$$\exists\alpha > 0, \exists\theta > 0, \forall(x, y) \in \Omega^2, \quad \|\mathbf{M}(x) - \mathbf{M}(y)\| \le \theta|x - y|^\alpha.$$

If $\mathbf{M} \in W^{1,p}(\Omega)$ and $p > n$ then, according to Sobolev embedding Theorem, condition (H) is satisfied with $\alpha = 1 - \frac{n}{p}$. Moreover, in [1], for a given $g \in L^\infty(\Omega)$, we have proposed a construction of $\mathbf{M}$ which ensures that this metric belongs to $W^{1,p}(\Omega)$. Hölder-regularity condition implies *Ellipticity* as follows.

Proposition 1.1. *If (H) is satisfied, then $\mathbf{M}$ satisfies **Ellipticity condition** (E):*

$$\exists\lambda > 0, \exists\Lambda > 0, \forall(x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \lambda|\mathbf{v}|^2 \le \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \le \Lambda|\mathbf{v}|^2. \tag{1.1}$$

Proof. As $\mathbf{M}(x)$ is symmetric positive definite for any $x \in \Omega$, we have

$$\forall x \in \Omega, \exists\lambda(x) > 0, \exists\Lambda(x) > 0, \forall\mathbf{v} \in \mathbb{R}^n, \quad \lambda(x)|\mathbf{v}|^2 \le \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \le \Lambda(x)|\mathbf{v}|^2.$$

According to (H), $\mathbf{M}$ is uniformly continuous, we may extend it in a continuous way to $\overline{\Omega}$. As Ω is bounded, then $\overline{\Omega}$ is compact and the previous inequalities remain true with (λ, Λ) independent of $x \in \Omega$. □

Definition 1.3. *For* $\mathbf{M} : \Omega \to S_n^+(\mathbb{R})$ *fixed, we define the* ***associated metric*** *as follows*

$$\forall (x, \mathbf{v}) \in \Omega \times \mathbb{R}^n, \quad \phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}(x)\mathbf{v}, \mathbf{v}\rangle^{\frac{1}{2}} \tag{1.2}$$

and the ***associated integrated distance*** *as*

$$\begin{array}{rcl} \forall (x,y) \in \Omega^2, \quad d_\phi(x,y) & = & \inf\left\{ \int_0^1 \phi(\gamma,\dot\gamma)\,dt \colon \begin{array}{l} \gamma \in W^{1,1}([0;1];\Omega), \\ \gamma(0)=x, \gamma(1)=y \end{array} \right\}, \\ \forall J \subset \Omega, \forall x \in \Omega, \quad d_\phi^J(x) & = & \inf\left\{ d_\phi(x,y) \colon y \in J \right\}. \end{array} \tag{1.3}$$

A straightforward consequence of [5], Theorem 3.2, is the following result.

Theorem 1.1. *Let* $J \subset \Omega$ *be a closed set. Then, we have*

$$\langle \mathbf{M}(x)\nabla d_\phi^J(x), \nabla d_\phi^J(x)\rangle = 1$$

at each point $x \in \Omega \setminus J$ *where* d_ϕ^J *is differentiable.*

1.3 Functionals defined on measures

Now, let $f : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ be a sublinear function with respect to the second variable, that is:

i) $\forall (x, \mathbf{v}_1, \mathbf{v}_2) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^n, \qquad f(x, \mathbf{v}_1 + \mathbf{v}_2) \leq f(x, \mathbf{v}_1) + f(x, \mathbf{v}_2),$

ii) $\forall (x, \mathbf{v}, t) \in \Omega \times \mathbb{R}^n \times \mathbb{R}^+, \qquad f(x, t\mathbf{v}) = tf(x, \mathbf{v}).$

Suppose that μ_1 is a Radon measure and μ_2 is a vectorial Radon measure on Ω. According to Besicovitch derivation theorem (see [6])

$$\lim_{r\to 0} \frac{\mu_2(B(x,r))}{\mu_1(B(x,r))}$$

exists and is finite for μ_1 almost every x, we denote by $\frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}(x)$ this limit when it exists. We recall that μ_2 is absolutely continuous with respect to μ_1 if $\mu_2(A) = 0$ whenever $\mu_1(A) = 0$. When this holds, we write $\mu_2 \ll \mu_1$. We consider the convex functional defined on the space $\mathcal{M}(\Omega; \mathbb{R}^n)$ by

$$\Phi : \mu_2 \in \mathcal{M}(\Omega; \mathbb{R}^n) \mapsto \int_\Omega f\left(x, \frac{\mathrm{d}\mu_2}{\mathrm{d}\mu_1}\right) \mathrm{d}\mu_1, \tag{1.4}$$

where μ_1 is a positive measure such that $\mu_2 \ll \mu_1$. It is shown in [7] that the integral in (1.4) does not depend on the choice of μ_1. For that reason, we will write it in the condensed form

$$\Phi(\mu_2) = \int_\Omega f\,(x, \mu_2).$$

We give a variant of the coarea formula extended to the sublinear functionals which can be found in [8].

Proposition 1.2. *Let* $\Phi(x, s, v)$ *a Borel function of* $\Omega \times \mathbb{R} \times \mathbb{R}^n$ *which is sublinear in* v*. Let* p *be a Lipschitz continuous function on* Ω *and denote, for* $t > 0$, $S_t = \{x \in \Omega; p(x) < t\}$*. Then, for almost all* $t \in \mathbb{R}$, S_t *belongs to* $BV(\Omega)$ *and we have*

$$\int_\Omega \Phi(x, p, Dp)\,dx = \int_\mathbb{R} dt \int_\Omega \Phi(x, t, D\mathbf{1}_{S_t}).$$

1.4 Functional spaces

We denote by

$$\begin{cases} B_r^+(x,\nu) = \{y \in B(x,r) \colon \langle y-x, \nu\rangle > 0\}, \\ B_r^-(x,\nu) = \{y \in B(x,r) \colon \langle y-x, \nu\rangle < 0\}, \end{cases}$$

the two half balls contained in the ball $B(x,r) \subset \mathbb{R}^n$ determined by $\nu \in \mathbb{S}^{n-1}$.

Definition 1.4. *Let $u \in L^1(\Omega)$ and $x \in \Omega$. We say that x is an approximate jump point of u if there exist $a, b \in \mathbb{R}$ and $\nu \in \mathbb{S}^{n-1}$ such that $a \neq b$ and*

$$\lim_{r \to 0^+} \fint_{B_r^+(x,\nu)} |u(y) - a| \, dy = 0, \quad \lim_{r \to 0^+} \fint_{B_r^-(x,\nu)} |u(y) - b| \, dy = 0.$$

The set of approximate jump points is denoted by J_u. The triplet (a, b, ν), uniquely determined up to a permutation of (a, b) and a change of sign of ν, is denoted by $(u^+(x), u^-(x), \nu_u(x))$.

We say that $u \in \mathbb{B}(\Omega)$ belongs to the space of *functions with bounded variation*, $\mathrm{BV}(\Omega)$, if $u \in \mathrm{L}^1(\Omega)$ and its derivative Du, in the sense of the distributions, is a Radon measure. According to [9], we have the following structure Theorem for the jump set of a BV function.

Theorem 1.2. *Let u be a given function in $BV(\Omega)$. There exists a countable family $(C_i)_{i \in \mathbb{N}}$ of compact $\mathcal{C}^1$-hypersurfaces such that*

$$J_u = \mathcal{N} \cup \left(\bigcup_{i \in \mathbb{N}} C_i \right),$$

where $\mathcal{H}^{n-1}(\mathcal{N}) = 0$.

We say that $u \in BV(\Omega)$ is a *special function with bounded variation* and we write $u \in SBV(\Omega)$, if the Cantor part of its derivative is zero, we obtain:

$$Du = \nabla u \mathcal{L}^n + (u^+ - u^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u,$$

where ∇u is the density of Du with respect to the Lebesgue measure $\mathcal{L}^n$ and $\mathcal{H}^{n-1} \llcorner J_u$ the restriction of the Hausdorff measure to the jump set.

We have the following chain rule for $\mathrm{SBV}(\Omega)$ (Theorem 3.99 in [6]).

Theorem 1.3. *Let $u \in SBV(\Omega)$ and let $f : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function. Then, $v = f \circ u$ belongs to $SBV(\Omega)$ and*

$$Dv = f'(u)\nabla u \mathcal{L}^n + (f(u^+) - f(u^-))\nu_u \mathcal{H}^{n-1} \llcorner J_u. \tag{1.5}$$

The following is a straightforward consequence of [6], Corollary 3.89.

Proposition 1.3. *Let Ω be an open and bounded domain satisfying (R) (1.1) and $\Omega' = U \cup \Omega$. For $u \in BV(\Omega)$, we consider an extension in Ω' by the following way*

$$\forall x \in U \setminus \overline{\Omega}, \quad u(x) = u(\varphi^{-1}(x)), \tag{1.6}$$

Then, we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

To establish our results, we need *slicing* tools.

Definition 1.5. *Let $\nu \in \mathbb{S}^{n-1}$ be fixed. We denote by Π_ν the hyperplane*

$$\{x \in \mathbb{R}^n \colon x \cdot \nu = 0\}.$$

If $x \in \Pi_\nu$, we set

$$\begin{array}{rcl} \Omega_x & = & \{t \in \mathbb{R} \colon x + t\nu \in \Omega\}, \\ \Omega_\nu & = & \{x \in \Pi_\nu \colon \Omega_x \neq \emptyset\}. \end{array}$$

For any function u defined on Ω and any $x \in \Omega_\nu$, we set

$$\begin{array}{rcl} (u)_x\colon \Omega_x & \to & \mathbb{R} \\ t & \to & u(x+t\nu) \end{array}$$

The following Theorem is proved in [11].

Theorem 1.4. *Let $u \in L^\infty(\Omega)$ be a function such that, for all $\nu \in \mathbb{S}^{n-1}$,*

i) $(u)_x \in SBV(\Omega_x)$ *for $\mathcal{H}^{n-1}$ a.e. $x \in \Omega_\nu$,*

ii) $\displaystyle\int_{\Omega_\nu} \left[\int_{\Omega_x} |\nabla (u)_x| dt + \mathcal{H}^0(J_{(u)_x}) \right] d\mathcal{H}^{n-1}(x) < +\infty;$

then, $u \in SBV(\Omega)$ and $\mathcal{H}^{n-1}(J_u) < +\infty$. Conversely, let $u \in SBV(\Omega) \cap L^\infty(\Omega)$ be such that $\mathcal{H}^{n-1}(J_u) < +\infty$. Then i) and ii) are satisfied. Moreover, we have

iii) $\langle \nabla u(x+t\nu), \nu\rangle = \nabla (u)_x(t)$, *for a.e. $t \in \Omega_x$ and $\mathcal{H}^{n-1}$-a.e. $x \in \Omega_\nu$,*

iv) $\displaystyle\int_{J_u} \langle \nu_u, \nu\rangle d\mathcal{H}^{n-1}(x) = \int_{\Omega_\nu} \mathcal{H}^0(J_{(u)_x}) d\mathcal{H}^{n-1}(x).$

1.5 Regularity results for free discontinuity problems

We need some regularity results of J_u for u a minimizer of E. For that we recall the definitions of *anisotropic Minkowski content* and of *almost quasi minimizer.*

Definition 1.6. *The anisotropic Minkowski $(n-1)$-dimensional upper and lower content associated to the metric ϕ (1.2) are defined by*

$$\begin{aligned} \mathcal{M}^\star_{\mathbf{M}}(J) &= \limsup_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega\colon d_\phi^J(x) < \rho\})}{2\rho}, \\ \mathcal{M}_{\star\mathbf{M}}(J) &= \liminf_{\rho\to 0^+} \frac{\mathcal{L}^n(\{x \in \Omega\colon d_\phi^J(x) < \rho\})}{2\rho}. \end{aligned}$$

In case they are equal, we call their common value the $(n-1)$-dimensional anisotropic Minkowski content $\mathcal{M}_{\mathbf{M}}(J)$.

Definition 1.7. *For $\Lambda \geq 1$, $\alpha > 0$ and $c_\alpha > 0$, we say that $w \in SBV(U)$ is an $(\Lambda, \alpha, c_\alpha)$-almost-quasi minimizer of a free discontinuity problem, if there exists $\Lambda \geq 1$, $\alpha > 0$ and $c_\alpha \geq 0$ such that*

$$\begin{gathered} v \in SBV(U), \quad x \in U, \quad \overline{B(x,r)} \subset U, \quad [w \neq v] \subset B(x,r) \qquad \Rightarrow \\ \int_{B(x,r)} |\nabla w|^2 dx + \mathcal{H}^{n-1}(J_w \cap \overline{B(x,r)}) \leq \int_{B(x,r)} |\nabla v|^2 dx + \Lambda \mathcal{H}^{n-1}(J_v \cap \overline{B(x,r)}) + c_\alpha r^{n-1+\alpha}. \end{gathered} \tag{1.7}$$

The following result is proved in [10].

Theorem 1.5. *Let $u \in SBV(\Omega)$ be an almost quasi-minimizer of a free discontinuity problem (1.7), then we have*

$$\mathcal{M}_{\mathbf{M}}(J_u) = \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

For our study, we are interested in the following Corollary.

Corollary 1.1. *Let Ω be an open and bounded domain, $h \in L^\infty(\Omega)$, $\alpha > 0$ and $\tilde{v} \in SBV(\Omega)$ a minimizer of*

$$\left\{ E^{\alpha,h}(v) = \alpha \int_\Omega (v-h)^2 dx + \int_\Omega |\nabla v|^2 dx + \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} \colon v \in SBV(\Omega) \right\}.$$

Then, we have

$$\mathcal{M}_{\mathbf{M}}(J_{\tilde{v}}) = \int_{J_{\tilde{v}}} \langle \mathbf{M}\nu_{\tilde{v}}, \nu_{\tilde{v}}\rangle^{\frac{1}{2}} d\mathcal{H}^{n-1}.$$

The proof of this Corollary is given in Appendix 4.1.

1.6 The functionals, their domains and the main result

Formally, we define the functionals $E(u)$ and $E_\varepsilon(u,z)$ as

$$\begin{aligned} E(u) &= \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 \mathrm{d}x + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\ E_\varepsilon(u,z) &= \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z\rangle + \frac{z^2}{4\varepsilon} \right) \mathrm{d}x. \end{aligned}$$

As in [4], the function $z : \Omega \to [0;1]$ plays the role of *control* of the gradient of u. We need to introduce a domain for E_ε that ensures the existence of a minimizer. If $u, z \in \mathrm{W}^{1,2}(\Omega)$ this functional is well defined. However, the coefficient $(1-z^2)^2$ removes the coercivity with respect to u and then the existence result can not be achieved according to the Sobolev norm. If, by addition, u is bounded, we have

$$\begin{aligned} |\nabla(u(1-z^2))|^2 &= |\nabla u(1-z^2) - 2uz\nabla z|^2, \\ &\leq 2|\nabla u|^2(1-z^2)^2 + 4\|u\|_{L^\infty(\Omega)}|\nabla z|^2. \end{aligned}$$

According to Ellipticity condition 1.1, it gives

$$\int_\Omega |\nabla(u(1-z^2))|^2 \mathrm{d}x \leq \left(2 + \frac{4\|u\|_{L^\infty(\Omega)}}{\lambda\varepsilon}\right) E_\varepsilon(u,z)$$

For that, it is natural to set

$$\mathcal{D}_n(\Omega) = \left\{(u,z) \colon u \in \mathbb{B}(\Omega),\ z \in \mathrm{W}^{1,2}(\Omega;[0;1]),\ \forall N \in \mathbb{N} \quad \overline{u}^N(1-z^2) \in \mathrm{W}^{1,2}(\Omega)\right\}, \tag{1.8}$$

where $\overline{u}^N$ is the truncated function defined by

$$\forall x \in \Omega, \quad \overline{u}^N(x) = \begin{cases} -N & \text{if } u(x) \leq -N, \\ u(x) & \text{if } |u(x)| \leq N, \\ N & \text{if } u(x) \geq N. \end{cases} \tag{1.9}$$

Assuming $(u,z) \in \mathcal{D}_n(\Omega)$ does not ensure that $u \in \mathrm{W}^{1,2}(\Omega)$ and ∇u can not be defined as the gradient of u in the Sobolev sense. However, we can define ∇u in the following sense.

Definition 1.8. *Let $u \in L^1(\Omega)$ and $x \in \Omega$ a Lebesgue point of u; we say that u is approximately differentiable at x if there exists $L \in \mathbb{R}^n$ such that*

$$\lim_{r \to 0^+} \fint_{B(x,r)} \frac{|u(y) - u(x) - \langle L, y-x\rangle|}{r} dy = 0. \tag{1.10}$$

If u is approximately differentiable at x then L, uniquely determined by (1.10), is called the approximate differential of u at x.

The following ensures that E_ε is well defined in $\mathcal{D}_n(\Omega)$.

Proposition 1.4. *If $(u,z) \in \mathcal{D}_n(\Omega)$, then u is approximately differentiable in $\{x \in \Omega \colon z(x) \neq 1\}$ and z is approximately differentiable in Ω.*

Proof. As Ω is open and bounded then $\mathrm{W}^{1,2}(\Omega) \subset BV(\Omega)$. According to Calderon-Zygmund (see [6], theorem 3.83), any function $u \in BV(\Omega)$ is approximately differentiable at almost every point $x \in \Omega$. So, if $(u,z) \in \mathcal{D}_n(\Omega)$, then z and $\overline{u}^N(1-z^2)$ are approximately differentiable almost everywhere. The following properties are straightforward consequences of Definition 1.10

- if v_1, v_2 are approximately differentiable almost everywhere and $v_1 \in L^\infty(\Omega)$, then $v_1 v_2$ is approximately differentiable almost everywhere;
- if v_2 is approximately differentiable almost everywhere, then ${v_2}^{-1}$ is also approximately differentiable almost everywhere in $\{x\colon v_2(x) \neq 0\}$ (Proposition 3.71 in [6]).

We deduce that $\overline{u}^N$ is approximately differentiable almost everywhere in $\{x\colon z(x) \neq 1\}$. This is true for any $N \in \mathbb{N}$, so this is also true for u. □

The main Theorem we prove in this paper is the following.

Theorem 1.6. *Assuming conditions* (R) *and* (H) *(defined in 1.1 and 1.2), let* $E : \mathbb{B}(\Omega) \to [0; +\infty]$ *defined as*

$$E(u) = \begin{cases} \displaystyle\int_\Omega (u-g)^2 dx + \int_\Omega |\nabla u|^2 dx + \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} d\mathcal{H}^{n-1} & \text{if } u \in SBV(\Omega), \\ +\infty & \text{otherwise} \end{cases}$$

and $E_\varepsilon : \mathbb{B}(\Omega) \times \mathbb{B}(\Omega) \to [0; +\infty]$ *defined as*

$$E_\varepsilon(u,z) = \begin{cases} \displaystyle\int_\Omega (u-g)^2 dx + \int_\Omega |\nabla u|^2 (1-z^2)^2 dx + \int_\Omega \left(\varepsilon \langle \mathbf{M}\nabla z, \nabla z \rangle + \frac{z^2}{4\varepsilon} \right) dx & \text{if } (u,z) \in \mathcal{D}_n(\Omega), \\ +\infty & \text{otherwise.} \end{cases}$$

Then, the following assertions are satisfied.

i) For any $\varepsilon > 0$*,* E_ε *admits a minimizer, denoted by* $(u_\varepsilon, z_\varepsilon) \in \mathcal{D}_n(\Omega)$*. Moreover, we can assume that* $u_\varepsilon(x) = g(x)$ *on* $\{x \in \Omega\colon z_\varepsilon(x) = 1\}$*.*

ii) For any $(\varepsilon_k)_k$ *converging to* 0^+*, there exists a subsequence, still denoted by* $(\varepsilon_k)_k$*, and* $u \in SBV(\Omega)$ *such that* $(u_{\varepsilon_k}, z_{\varepsilon_k})_k$ *converges to* $(u, 0)$ *almost everywhere and* u *is a minimizer of* E*.*

We denote by

$$(\mathcal{P}) : \quad \text{Min}\{E(u)\colon u \in \mathbb{B}(\Omega)\},$$

$$(\mathcal{P}_\varepsilon) : \quad \text{Min}\{E_\varepsilon(u,z)\colon (u,z) \in \mathbb{B}(\Omega) \times \mathbb{B}(\Omega)\},$$

the two minimization problem. Theorem 1.6 *i*) implies that, for $\varepsilon > 0$ fixed, $(\mathcal{P}_\varepsilon)$ is a well posed problem. Theorem 1.6 *ii*) implies that, up to the extraction of a subsequence, the minimizers of $(\mathcal{P}_\varepsilon)$ converge to a solution of $(\mathcal{P})$.

2 Existence result for $(\mathcal{P}_\varepsilon)$

We prove Theorem 1.6 *i*). For that, we follow the *direct method* of calculus of variations: first we show compactness of a minimizing sequence (Proposition 2.1), then we prove a lower semi-continuity result for the functional E_ε (Proposition 2.2). Theorem 1.6 *i*) is a straightforward consequence of these results.

2.1 Compactness

In this section, we prove the following.

Proposition 2.1. *Let* $\varepsilon > 0$ *be fixed. There exists* (u_k, z_k) *a minimizing sequence of* E_ε *such that* $(u_k)_k$ *is a bounded sequence of* $L^\infty(\Omega)$*,* $(u_k, z_k)_k$ *converges almost everywhere to* $(u,z) \in \mathcal{D}_n(\Omega)$ *and* $u(x) = g(x)$ *on* $\{x \in \Omega\colon z(x) = 1\}$*.*

We need the following Lemma which is proved in Appendix 4.2.

Lemma 2.1. *For $(u,z) \in \mathcal{D}_n(\Omega)$ and $\nu \in \mathbb{S}^{n-1}$ fixed, we have $((u)_x, z_x) \in \mathcal{D}_1(\Omega_x)$ for $\mathcal{H}^{n-1}$-almost every $x \in \Omega_\nu$ and*

$$\begin{array}{rcl} \nabla(u)_x(t) &=& \langle \nabla u(x+t\nu), \nu\rangle, \\ \nabla z_x(t) &=& \langle \nabla z(x+t\nu), \nu\rangle, \end{array}$$

for almost every $t \in \Omega_x \setminus \{s\colon z(x+s\nu)=1\}$.

Now, we prove Proposition 2.1.

Proof. As $E_\varepsilon \geq 0$, there exists a minimizing sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$. We fix $N \geq \|g\|_{L^\infty(\Omega)}$ and we consider the truncated functions $(\overline{u}_k^N)_k$ defined in (1.9). As $(u_k, z_k) \in \mathcal{D}_n(\Omega)$, we have $\overline{u}_k^N(1-z_k^2) \in W^{1,2}(\Omega)$. As Ω is bounded, then $\mathrm{W}^{1,2}(\Omega) \subset \mathrm{SBV}(\Omega)$. According to Calderón-Zygmund Theorem (3.83 in [6]), $\overline{u}_k^N(1-z_k^2)$ is approximately differentiable almost everywhere. For the same reasons, $1-z_k^2$ is also approximately differentiable almost everywhere. According to Proposition 3.71 in [6], we deduce that $\overline{u}_k^N$ is approximately differentiable almost everywhere in $\{x\colon z_k(x) \neq 1\}$. Moreover, $\nabla \overline{u}_k^N(x) = 0$ almost everywhere in $\{x\colon |\overline{u}_k^N(x)| = N\}$ and $\nabla \overline{u}_k^N(x) = \nabla u_k(x)$ almost everywhere in $\{x\colon |\overline{u}_k^N(x)| < N\}$ (Proposition 3.73 in [6]), it gives

$$\forall k \in \mathbb{N}, \quad \int_\Omega |\nabla \overline{u}_k^N|^2 (1-z_k^2)^2 \mathrm{d}x \leq \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x. \tag{2.1}$$

so $E_\varepsilon(\overline{u}_k^N, z_k) \leq E_\varepsilon(u_k, z_k)$ and then $(\overline{u}_k^N, z_k)_k$ is also a minimizing sequence. According to Ellipticity condition 1.1, we have

$$\int_\Omega |\nabla z_k|^2 \mathrm{d}x + \int_\Omega z_k^2 \mathrm{d}x \leq \left(\frac{1}{\lambda\varepsilon} + 4\varepsilon\right) E_\varepsilon(u_k, z_k),$$

and then $(z_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(z_k)_k$, which converges almost everywhere to $z \in \mathrm{W}^{1,2}(\Omega)$. moreover, as $(z_k)_k$ takes its values almost everywhere in $[0;1]$, then z takes also its values in $[0;1]$. For $w_k = \overline{u}_k^N(1-z_k^2)$, we have

$$\int_\Omega |\nabla w_k|^2 \mathrm{d}x + \int_\Omega w_k^2 \mathrm{d}x \leq 2\int_\Omega |\nabla \overline{u}_k^N|^2 (1-z_k^2)^2 \mathrm{d}x + 2N^2 \int_\Omega |\nabla z_k|^2 \mathrm{d}x + N^2 \int_\Omega (1-z_k^2)^2 \mathrm{d}x$$

and then $(w_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(\Omega)$. So, there exists a subsequence, still denoted by $(w_k)_k$, which converges almost everywhere to $w \in \mathrm{W}^{1,2}(\Omega)$. In particular, $(\overline{u}_k^N(x))_k$ converges for almost every $x \in \{y\colon z(y) \neq 1\}$ to $u(x)$. We set $u(x) = g(x)$ for $x \in \{y\colon z(y) = 1\}$. This construction ensures that $(u,z) \in \mathcal{D}_n(\Omega)$.

□

2.2 Lower semi-continuity

We prove the following.

Proposition 2.2. *If $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converges almost everywhere to $(u,z) \in \mathcal{D}_n(\Omega)$ and $(u_k)_k$ is a bounded sequence of $L^\infty(\Omega)$, then*

$$\liminf_{k\to\infty} E_\varepsilon(u_k, z_k) \geq E_\varepsilon(u,z).$$

Proof. Fatou Lemma gives

$$\liminf_{k\to\infty} \left(\int_\Omega (u_k - g)^2 \mathrm{d}x + \int_\Omega \frac{z_k^2}{4\varepsilon} \mathrm{d}x \right) \geq \int_\Omega (u-g)^2 \mathrm{d}x + \int_\Omega \frac{z^2}{4\varepsilon} \mathrm{d}x,$$

So, to show Proposition 2.2, it suffices to prove that

$$\liminf_{k\to\infty}\int_\Omega \varepsilon\langle \mathbf{M}\nabla z_k,\nabla z_k\rangle \mathrm{d}x \geq \int_\Omega \varepsilon\langle \mathbf{M}\nabla z,\nabla z\rangle \mathrm{d}x \tag{2.2}$$

and

$$\liminf_{k\to\infty}\int_\Omega |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x \geq \int_\Omega |\nabla u|^2(1-z^2)^2 \mathrm{d}x. \tag{2.3}$$

Proof of (2.2)

As the application

$$\begin{array}{ccc} \mathrm{W}^{1,2}(\Omega) & \to & \mathrm{L}^2(\Omega;\mathbb{R}^n), \\ z & \to & \nabla z \end{array}$$

is continuous for the strong topology, it remains to prove that the application

$$\begin{array}{ccc} \mathrm{L}^2(\Omega;\mathbb{R}^n) & \to & \mathbb{R}, \\ Z & \to & \int_\Omega \langle \mathbf{M}Z,Z\rangle \mathrm{d}x \end{array}$$

is lower semi-continuous for the weak topology of $\mathrm{L}^2(\Omega;\mathbb{R}^n)$. Let $(Z_k)_k \subset \mathrm{L}^2(\Omega;\mathbb{R}^n)$ be weakly convergent to $Z \in \mathrm{L}^2(\Omega;\mathbb{R}^n)$. We set

$$\begin{array}{ccc} L:\mathrm{L}^2(\Omega;\mathbb{R}^n) & \to & \mathbb{R}, \\ U & \to & \int_\Omega \langle \mathbf{M}Z,U\rangle \mathrm{d}x \end{array}$$

According to Ellipticity condition (1.1), $L \in (\mathrm{L}^2(\Omega;\mathbb{R}^n))'$ and then $(L(Z_k))_k$ converges to $L(Z)$. Moreover, for k fixed, the following polynomial function is positive

$$t \to \int_\Omega \langle \mathbf{M}(Z+tZ_k), Z+tZ_k\rangle \mathrm{d}x.$$

Thus, its discriminant is negative and we deduce the following *anisotropic Cauchy-Schwarz inequality*

$$\int_\Omega \langle \mathbf{M}Z,Z_k\rangle \mathrm{d}x \leq \left(\int_\Omega \langle \mathbf{M}Z,Z\rangle \mathrm{d}x\right)^{\frac{1}{2}} \left(\int_\Omega \langle \mathbf{M}Z_k,Z_k\rangle \mathrm{d}x\right)^{\frac{1}{2}}.$$

As $(L(Z_k))_k$ converges to $L(Z)$, passing through the $\liminf$ in the previous inequality yields

$$\int_\Omega \langle \mathbf{M}Z,Z\rangle \mathrm{d}x \leq \left(\int_\Omega \langle \mathbf{M}Z,Z\rangle \mathrm{d}x\right)^{\frac{1}{2}} \liminf_{k\to\infty} \left(\int_\Omega \langle \mathbf{M}Z_k,Z_k\rangle \mathrm{d}x\right)^{\frac{1}{2}}$$

and then we may conclude the *Proof of (2.2)* by taking $Z_k = \nabla z_k, Z = \nabla z$ in the previous inequality

$$\int_\Omega \langle \mathbf{M}\nabla z,\nabla z\rangle \mathrm{d}x \leq \liminf_{k\to\infty}\int_\Omega \langle \mathbf{M}\nabla z_k,\nabla z_k\rangle \mathrm{d}x.$$

Proof of (2.3)

We first consider the one-dimensional case $n=1$ and then by a slicing argument we get the lower semi-continuity for the general case $n \geq 1$. Let $A \subset \{x \in \Omega\colon z(x) < 1\}$ be an open and relatively compact subset of $\Omega \subset \mathbb{R}$. As $(z_k)_k$ weakly converges to z in $W^{1,2}(\Omega)$, then $(z_k)_k$ uniformly converges to z. In particular, there exists $\delta > 0$ and $k_0 \in \mathbb{N}$ such that

$$k \geq k_0 \quad \Rightarrow \quad A \subset \{x \in \Omega\colon z_k(x) \leq 1-\delta\}.$$

Thus, we have

$$\forall k \geq k_0, \quad \int_A |\nabla u_k|^2 \mathrm{d}x \leq \frac{1}{1-(1-\delta)^2} E_\varepsilon(u_k, z_k)$$

and then we deduce that $(u_k)_k$ is a bounded sequence of $\mathrm{W}^{1,2}(A)$. As $u_k(1-z_k^2)$ converges almost everywhere to $u(1-z^2)$ in Ω, there exists a subsequence, still denoted by $(u_k)_k$, which weakly converges to u in $\mathrm{W}^{1,2}(A)$. In particular, $(\nabla u_k)_k$ weakly converges to ∇u in $\mathrm{L}^2(A)$. For $\xi \in \mathrm{L}^2(\Omega)$, we decompose

$$\int_A \xi \left[\nabla u_k(1-z_k^2) - \nabla u(1-z^2)\right] \mathrm{d}x = \int_A \xi \nabla u_k(z^2 - z_k^2)\mathrm{d}x + \int_A \xi \nabla u_k(1-z^2)\mathrm{d}x + \int_A \xi \nabla u(z^2 - z_k^2)\mathrm{d}x. \tag{2.4}$$

As $(1-z^2)\xi \in \mathrm{L}^2(A)$ and $(\nabla u_k)_k$ weakly converges to ∇u in $\mathrm{L}^2(A)$, then we have

$$\int_A \xi \nabla u_k(1-z^2)\mathrm{d}x \to \int_A \xi \nabla u(1-z^2)\mathrm{d}x. \tag{2.5}$$

Moreover, we have

$$\int_A \xi \nabla u_k(z - z_k)\mathrm{d}x \leq \|\xi\|_{L^2(A)} \|\nabla u_k\|_{L^2(A)} \|z^2 - z_k^2\|_{L^\infty(A)}$$

and

$$\int_A \xi \nabla u(z - z_k)\mathrm{d}x \leq \|\xi\|_{L^2(A)} \|\nabla u\|_{L^2(A)} \|z^2 - z_k^2\|_{L^\infty(A)}.$$

As a weakly convergent sequence is bounded, then $(\nabla u_k)_k$ is bounded in $\mathrm{L}^2(A)$ and we deduce that

$$\int_A \xi \nabla u_k(z^2 - z_k^2)\mathrm{d}x \to 0, \qquad \int_A \xi \nabla u(z^2 - z_k^2)\mathrm{d}x \to 0. \tag{2.6}$$

According to (2.4), (2.5) and (2.6), we get

$$\int_A \xi \nabla u_k(1-z_k^2)\mathrm{d}x \to \int_A \xi \nabla u(1-z^2)\mathrm{d}x$$

and then $(\nabla u_k(1-z_k^2))_k$ weakly converges to $\nabla u(1-z^2)$ in $\mathrm{L}^2(A)$. As the norm is lower semi-continuous, we deduce

$$\begin{aligned}
\int_A |\nabla u|^2(1-z^2)^2 \mathrm{d}x &\leq \liminf_{k\to\infty} \int_A |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \\
&\leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.
\end{aligned}$$

Passing to the limit $A \uparrow \{x \in \Omega \colon z(x) < 1\}$ gives

$$\int_\Omega |\nabla u|^2 (1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.$$

We generalize this result to the dimension $n \geq 1$. With the notation $(u)_x$ introduced in (1.5), using the previous result obtained in dimension 1, Lemma 2.1 and Fatou Lemma, give

$$\begin{aligned}
\int_A |\langle \nabla u, \nu\rangle|^2 (1-z^2)^2 \mathrm{d}x &= \int_{A_\nu} \int_{A_x} |\nabla (u)_x(t)|^2 (1 - z_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\
&\leq \int_{A_\nu} \liminf_{k\to\infty} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1 - (z_k)_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\
&\leq \liminf_{k\to\infty} \int_{A_\nu} \int_{A_x} |\nabla (u_k)_x(t)|^2 (1 - (z_k)_x(t)^2)^2 \mathrm{d}t\mathrm{d}x, \\
&\leq \liminf_{k\to\infty} \int_{A_\nu} \int_{A_x} |\langle \nabla u_k(x+t\nu), \nu\rangle|^2 (1 - z_k(x+t\nu)^2)^2 \mathrm{d}t\mathrm{d}x, \\
&\leq \liminf_{k\to\infty} \int_A |\langle \nabla u_k, \nu\rangle|^2 (1-z_k^2)^2 \mathrm{d}x, \\
&\leq \liminf_{k\to\infty} \int_A |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x,
\end{aligned}$$

for any open set $A \subset \Omega$ and every $\nu \in \mathbb{S}^{n-1}$. The function $x \to \frac{\nabla u(x)}{|\nabla u(x)|}$ is measurable in $U = \{x \in \Omega \colon z(x) \neq 1, \nabla u(x) \neq 0\}$. According to Lusin Theorem (1.45 of [6]), there exists an increasing sequence of compacts $(K_l)_l \subset U$ such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \to \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any $x \in K_l$, there exists $r > 0$ such that

$$y \in B(x,r) \quad \Rightarrow \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right| \leq \frac{1}{l}, \tag{2.7}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [6]), there exists a countable, pairwise disjoint collection of balls $(B_i)_{i \in I}$ satisfying (3.13) such that

$$\forall i \in I \quad B_i \subset \Omega, \qquad \mathcal{L}^n\left(K_l \setminus \bigcup_{i \in I} B_i\right) = 0.$$

For any $i \in I$, we fix $x_i \in B_i$ and we set $\nu_i = \frac{\nabla u(x_i)}{|\nabla u(x_i)|}$; then

$$\int_{B_i} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x \leq \liminf_{k \to \infty} \int_{B_i} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.$$

As $(B_i)_i$ is pairwise dijoint, we deduce

$$\begin{aligned} \int_{\cup_i B_i} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x &= \sum_{i \in I} \int_{B_i} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x, \\ &\leq \sum_{i \in I} \liminf_{k \to \infty} \int_{B_i} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \\ &\leq \liminf_{k \to \infty} \sum_{i \in I} \int_{B_i} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \\ &\leq \liminf_{k \to \infty} \int_{\cup_i B_i} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x, \\ &\leq \liminf_{k \to \infty} \int_{\Omega} |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x. \end{aligned}$$

Moreover, we have

$$\begin{aligned} \forall x \in B_i \cap K_l, \quad \left| |\nabla u|^2 - |\langle \nabla u, \nu_i \rangle|^2 \right| &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} \rangle^2 - \langle \nabla u, \nu_i \rangle^2 \right|, \\ &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} - \nu_i \rangle \langle \nabla u, \frac{\nabla u}{|\nabla u|} + \nu_i \rangle \right|, \\ &\leq \frac{2}{l} |\nabla u|^2. \end{aligned}$$

It gives

$$\int_{B_i \cap K_l} |\nabla u|^2 (1-z^2)^2 \mathrm{d}x \leq \frac{l}{l-2} \int_{B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x.$$

As $\mathcal{L}^n(K_l \setminus \cup_i B_i) = 0$ and $(B_i)_i$ is pairwise disjoint, we get

$$\begin{aligned}
\int_{K_l} |\nabla u|^2(1-z^2)^2 \mathrm{d}x &= \sum_i \int_{B_i \cap K_l} |\nabla u|^2(1-z^2)^2 \mathrm{d}x, \\
&\leq \frac{l}{l-2} \sum_i \int_{B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x, \\
&\leq \frac{l}{l-2} \int_{\cup_i B_i \cap K_l} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x, \\
&\leq \frac{l}{l-2} \int_{\cup_i B_i} |\langle \nabla u, \nu_i \rangle|^2 (1-z^2)^2 \mathrm{d}x, \\
&\leq \frac{l}{l-2} \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2 (1-z_k^2)^2 \mathrm{d}x.
\end{aligned}$$

As $(K_l)_l$ is an increasing sequence such that $\mathcal{L}^n(U \setminus K_l) \to 0$, passing to the limit $l \to \infty$ gives

$$\int_U |\nabla u|^2(1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x$$

and we may conclude

$$\int_\Omega |\nabla u|^2(1-z^2)^2 \mathrm{d}x \leq \liminf_{k\to\infty} \int_\Omega |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x.$$

□

3 Approximation result for $\varepsilon \to 0^+$

This section is dedicated to the proof of Theorem 1.6 *ii)*. For that, we introduce the following Γ-convergence result (see [6], Definition 6.12 for a formal definition).

Theorem 3.1. *Assuming conditions (R) and (H) and $(\varepsilon_k)_k$ converging to 0^+, we have*

i) if $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere, then

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_k, z_k) \geq E(u); \tag{3.1}$$

ii) for any $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$, $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere and

$$\limsup_{k\to\infty} E_{\varepsilon_k}(u_k, z_k) \leq E(u). \tag{3.2}$$

3.1 The inequality for the *lower* Γ-limit

We now prove the first inequality of Γ-convergence (3.1). Let $u \in \mathbb{B}(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere. In the sequel, we emphasize on the domain of the function: for U an open subset of Ω, we adopt the following notation

$$\begin{aligned}
F(u; U) &= \int_U |\nabla u|^2 \mathrm{d}x + \int_{J_u \cap U} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\
F_{\varepsilon_k}(u_k, z_k; U) &= \int_U |\nabla u_k|^2(1-z_k^2)^2 \mathrm{d}x + \int_U \left(\varepsilon_k \langle \mathbf{M}\nabla z_k, \nabla z_k \rangle + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}x,
\end{aligned}$$

Fatou Lemma yields

$$\liminf_{k\to\infty}\int_\Omega (u_k-g)^2\mathrm{d}x \geq \int_\Omega (u-g)^2\mathrm{d}x$$

and then it suffices to prove that $\liminf F_{\varepsilon_k}(u_k, z_k;\Omega)\geq F(u;\Omega)$.

Let f be a function defined on open sets, we adopt the following vocabulary

- f is superaddditive if

$$A\cap B=\emptyset \qquad \Rightarrow \qquad f(A\cup B)\geq f(A)+\ f(B), \tag{3.3}$$

- f is non decreasing if

$$A\subset B \qquad \Rightarrow \qquad f(A)\leq f(B). \tag{3.4}$$

We perform the proof in two steps: the first step deals with dimension 1. The second generalizes it for dimension $n\geq 2$.

3.1.1 The one-dimensional case

In this section, we assume that $\Omega = I$ is an open interval and the metric $\mathbf{M}$ is simply a constant $m>0$. To avoid confusion, we denote the approximating functional by

$$G_\varepsilon(u,z;I)=\int_I |\nabla u(t)|^2(1-z(t)^2)^2\mathrm{d}t+\int_I\left(m\varepsilon|\nabla z(t)|^2+\frac{z(t)^2}{4\varepsilon}\right)\mathrm{d}t, \tag{3.5}$$

where the domain is

$$\mathcal{D}_1(I)=\{(u,z)\colon u\in\mathbb{B}(I), z\in \mathrm{W}^{1,2}(I;[0;1]), \forall N\in\mathbb{N}\quad \overline{u}^N(1-z^2)\in \mathrm{W}^{1,2}(I)\}.$$

We denote the lower Γ-limit, by

$$G_-(u;I)=\inf\left\{\liminf_{k\to\infty} G_{\varepsilon_k}(u_k,z_k;I)\right\},$$

where the inf is taken over all sequence $(u_k,z_k)_k\subset\mathcal{D}_1(I)$ such that (u_k,z_k) converges almost everywhere to $(u,0)$ in I. We need the following Lemma which proof is given in appendix 4.3.

Lemma 3.1. *Let $I\subset\mathbb{R}$ be an open interval, $J\subset I$ be a set with finite cardinal. We have*

$$u\in W^{1,2}(I\setminus J)\Rightarrow u\in SBV(I),\ \ J_u\subset J.$$

The main result of this subsection is given by the following.

Proposition 3.1. *Let $I\subset\mathbb{R}$ be an open interval and $u\in\mathbb{B}(I)$. If $G_-(u;I)<\infty$, then $u\in SBV(I)$ and*

$$\int_I|\nabla u(t)|^2dt+m^{\frac{1}{2}}\mathcal{H}^0(J_u\cap I)\leq G_-(u;I).$$

The proof of this Proposition consists in showing the two following Lemma.

Lemma 3.2. *If $u\in W^{1,2}(B_\eta(x))$, then we have*

$$G_-(u;B_\eta(x))\geq\int_{B_\eta(x)}|\nabla u(t)|^2dt.$$

Lemma 3.3. *If $u\notin W^{1,2}(B_\rho(x))$ for any $\rho\in]0;\eta[$, then we have*

$$\forall\rho\in]0;\eta[,\quad G_-(u;B_\rho(x))\geq m^{\frac{1}{2}}.$$

Suppose that Lemma 3.2 and 3.3 are proved, we deduce the Proposition 3.1.

Proof. We set
$$J = \left\{ x \in I \colon \forall \rho > 0,\ u \notin \mathrm{W}^{1,2}(B_\rho(x)) \right\}.$$

Let $\{x_1, \ldots, x_N\} \subset J$ and $\rho > 0$ be such that $\{B_\rho(x_i) \colon i = 1, \ldots, N\}$ is pairwise disjoint. According to Lemma 3.3 we have

$$\forall i \in \{1, \ldots, N\}, \quad G_-(u; B_\rho(x_i)) \geq m^{\frac{1}{2}}$$

and then
$$\sum_{i=1}^{N} G_-(u; B_\rho(x_i)) \geq N m^{\frac{1}{2}}.$$

As $G_-(u; \cdot)$ is superadditive, we have

$$G_-(u; \cup_{i=1}^{N} B_\rho(x_i)) \geq N m^{\frac{1}{2}}.$$

and $G_-(u; \cdot)$ is non decreasing, it gives

$$G_-(u; I) \geq N m^{\frac{1}{2}}.$$

As $G_-(u; \cdot) < +\infty$, the set J is finite. So, there exists $\rho > 0$ such that $\{B_\rho(x) \colon x \in J\}$ is pairwise disjoint. As $G_-(u;\ \cdot\ )$ is superadditive (3.3) and non decreasing (3.4), we have

$$\sum_{x \in J} G_-(u; B_\rho(x)) + G_-(u; I \setminus \cup_{x \in J} \overline{B_\rho(x)}) \leq G_-(u; I).$$

According to Lemma 3.2 and 3.3, it gives

$$\mathcal{H}^0(J) m^{\frac{1}{2}} + \int_{I \setminus \cup_{x \in J} \overline{B_\rho(x)}} |\nabla u(t)|^2 \mathrm{d}t \leq G_-(u; I).$$

Taking the limit $\rho \to 0^+$ yields

$$\mathcal{H}^0(J) m^{\frac{1}{2}} + \int_{I \setminus J} |\nabla u(t)|^2 \mathrm{d}t \leq G_-(u; I).$$

In particular $u \in \mathrm{W}^{1,2}(I \setminus J)$ and, according to Lemma 3.1, we get $u \in \mathrm{SBV}(I)$, $J_u \subset J$ and then

$$\mathcal{H}^0(J) m^{\frac{1}{2}} + \int_I |\nabla u(t)|^2 \mathrm{d}t \leq G_-(u; I).$$

□

Now, we prove lemma 3.2.

Proof. We can assume that $G_-(u; B_\eta(x)) < +\infty$, otherwise the result is ensured. By a diagonal extraction, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_1(B_\rho(x))$ converging almost everywhere to $(u, 0)$ and

$$G_{\varepsilon_k}(u_k, z_k; B_\rho(x)) \to G_-(u; B_\rho(x)).$$

As $G_-(u; B_\eta(x))$ is finite, there exists $C > 0$ such that

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} \left(\varepsilon_k |\nabla z_k|^2 + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}t \leq C. \tag{3.6}$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a^2 = \varepsilon_k |\nabla z_k|^2$ and $b^2 = \frac{z_k^2}{4\varepsilon_k}$ gives

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla z_k| z_k \mathrm{d}t \leq C. \tag{3.7}$$

We set $c_k = 1 - z_k^2$. As $z_k \in \mathrm{W}^{1,2}(B_\eta(x))$, then $c_k \in BV(B_\eta(x))$ and (3.7) is

$$\forall k \in \mathbb{N}, \quad \int_{B_\eta(x)} |\nabla c_k| \mathrm{d}t \leq 2C.$$

Coarea formula (see [6]) yields

$$\forall k \in \mathbb{N}, \quad \int_0^1 \mathcal{H}^0(\{y \in B_\eta(x) \colon c_k(y) = t\}) \mathrm{d}t \leq 2C. \tag{3.8}$$

Let $\sigma < 1$ in an arbitrary neighborhood of 1 and $\delta \in]0; \sigma[$ be fixed numbers. According to (3.8) and mean value theorem, there exits $\delta_k \in]\delta; \sigma[$ such that

$$\forall k \in \mathbb{N}, \quad \mathcal{H}^0(\{y \in B_\eta(x) \colon c_k(y) = \delta_k\}) \leq \frac{2C}{\sigma - \delta}. \tag{3.9}$$

We set $A_k = \{y \in B_\eta(x) \colon c_k(y) \geq \delta_k\}$. As $(\varepsilon_k)_k$ converges to 0, inequality (3.6) implies that $(z_k)_k$ converges to 0 and c_k to 1 almost everywhere. As $\delta_k < \sigma$ and $\sigma < 1$ then $(\mathcal{L}^1(A_k))_k$ converges to $\mathcal{L}^1(B_\eta(x))$.

Sobolev embedding theorem ensures that $\mathrm{W}^{1,2}(B_\eta(x)) \subset \mathcal{C}(B_\eta(x))$, so c_k is continuous and A_k is a countable union of closed intervals of $B_\eta(x)$. According to (3.9), this union is finite and its cardinality is uniformly bounded by N. For any k, there exits a disjoint family of closed intervals $(I_k^i)_{i=1\ldots N}$ such that

$$\begin{cases} A_k = \bigcup_{i=1}^N I_k^i, \\ \forall i \in \{1, \ldots, N-1\}, \max(I_k^i) < \min(I_k^{i+1}). \end{cases}$$

There exists a subsequence, still denoted by $(I_k^i)_{i=1\ldots N}$, such that $(\min(I_k^i))_k$ and $(\max(I_k^i))_k$ converge for any $i \in \{1, \ldots, N\}$. We set a_∞^i and b_∞^i the previous limits, $I_\infty^i =]a_\infty^i; b_\infty^i[$ and $A = \bigcup_{i=1}^N I_\infty^i$. As $(\mathcal{L}^1(A_k))_k$ converges to $\mathcal{L}^1(B_\eta(x))$, then A is a subset of full measure in $B_\eta(x)$.

Let O be an open subset such that $\overline{O} \subset A$. For k with a sufficiently large value, we have $O \subset A_k$ and then

$$\liminf_{k \to \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t \geq \liminf_{k \to \infty} \int_O |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t.$$

As z_k takes its values in $[0; 1]$, we get

$$\begin{aligned} \liminf_{k \to \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t &\geq \liminf_{k \to \infty} \int_O |\nabla u_k|^2 \delta_k^2 \mathrm{d}t, \\ &\geq \delta^2 \liminf_{k \to \infty} \int_O |\nabla u_k|^2 \mathrm{d}t. \end{aligned}$$

Moreover, there exists k_0 such that: $k \geq k_0 \Rightarrow O \subset A_k$, then we have

$$\forall x \in O \quad 1 - z_k^2 \geq \delta.$$

As $u_k(1 - z_k^2) \in W^{1,2}(\Omega)$ and $\sqrt{\delta} > 0$, we get $u_k \in \mathrm{W}^{1,2}(O)$ and the lower semi-continuity property of the Sobolev norm gives

$$\liminf_{k \to \infty} \int_O |\nabla u_k|^2 \mathrm{d}t \geq \int_O |\nabla u|^2 \mathrm{d}t.$$

As O is chosen arbitrary in A and A is of full measure in $B_\eta(x)$, it gives

$$\liminf_{k \to \infty} \int_{B_\eta(x)} |\nabla u_k|^2 (1 - z_k^2)^2 \mathrm{d}t \geq \delta^2 \int_{B_\eta(x)} |\nabla u|^2 \mathrm{d}t.$$

Letting δ to 1^-, it concludes the proof of lemma 3.2.

□

We prove lemma 3.3.

Proof. We can assume that $G_-(u;B_\rho(x)) < +\infty$ for any $\rho \in]0;\eta[$, otherwise the result is ensured. As $u \notin \mathrm{W}^{1,2}(B_\rho(x))$, there exists three sequences $(y_k^1)_{k\in\mathbb{N}}$, $(y_k^2)_{k\in\mathbb{N}}$ and $(y_k^3)_{k\in\mathbb{N}}$ such that:

$$\begin{cases} y_k^1 \to x, \;\; z_k(y_k^1) \to 0, \\ y_k^2 \to x, \;\; z_k(y_k^2) \to 1, \\ y_k^3 \to x, \;\; z_k(y_k^3) \to 0, \\ \forall k \in \mathbb{N}, \;\; y_k^1 < y_k^2 < y_k^3. \end{cases} \tag{3.10}$$

We have

$$G(u_k, z_k; B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} \left(\varepsilon_k m |\nabla z_k|^2 + \frac{z_k^2}{4\varepsilon_k} \right) \mathrm{d}t.$$

The inequality $a^2 + b^2 \geq 2ab$ gives:

$$G(u_k, z_k; B_\rho(x)) \geq \int_{x-\rho}^{x+\rho} m^{\frac{1}{2}} |\nabla z_k| z_k \,\mathrm{d}t.$$

As $[y_k^1, y_k^2] \subset B_\rho(x)$, we obtain:

$$G(u_k, z_k; B_\rho(x)) \geq \int_{y_k^1}^{y_k^3} m^{\frac{1}{2}} |\nabla z_k| z_k \,\mathrm{d}t.$$

We have

$$G(u_k, z_k; B_\rho(x)) \geq m^{\frac{1}{2}} \int_{y_k^1}^{y_k^2} |\nabla z_k(t)| z_k(t) \,\mathrm{d}t + m^{\frac{1}{2}} \int_{y_k^2}^{y_k^3} |\nabla z_k(t)| z_k(t) \,\mathrm{d}t.$$

Since $z_k \in \mathrm{W}^{1,2}(B_\eta(x))$, we may use the change of variable $s = z_k(t)$. This yields:

$$\begin{aligned} (\star)_k^2 \;&\geq\; m^{\frac{1}{2}} \int_{z_k(y_k^1)}^{z_k(y_k^2)} s \mathrm{d}s + m^{\frac{1}{2}} \int_{z_k(y_k^2)}^{z_k(y_k^3)} s \mathrm{d}s, \\ &\geq\; m^{\frac{1}{2}} \left(\frac{z_k^2(y_k^2) - z_k^2(y_k^1)}{2} + \frac{z_k^2(y_k^2) - z_k^2(y_k^3)}{2} \right) \end{aligned}$$

By assumption, we have $z_k(y_k^1) \to 0$, $z_k(y_k^2) \to 1$ and $z_k(y_k^3) \to 0$, so that we deduce:

$$\frac{z_k^2(y_k^2) - z_k^2(y_k^1)}{2} + \frac{z_k^2(y_k^2) - z_k^2(y_k^3)}{2} \to 1.$$

We can conclude :

$$\liminf_{k\to\infty} G(u_k, z_k; B_\rho(x)) \geq m^{\frac{1}{2}}.$$

□

3.1.2 Generalization to dimension $n \geq 2$

We give the proof of the first inequality of Γ-convergence (3.1) for $n \geq 2$.

Proof. Let $u \in \mathrm{SBV}(\Omega) \cap L^\infty(\Omega)$ and $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ converging almost everywhere to $(u, 0)$ such that $(u_k)_k$ is bounded in $\mathrm{L}^\infty(\Omega)$. We have to prove

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \Omega) \geq F(u; \Omega). \tag{3.11}$$

We assume that $\liminf F_{\varepsilon_k}(u_k, z_k; \Omega)$ is finite, otherwise the result is ensured.

First Step: There exists $C(\delta)$, also depending on the regularity parameters $(\lambda, \theta, \alpha)$ (Definition 1.2), such that

i) $C(\delta) \to 1$,

ii) for $A \subset \Omega$ open, $a \in A$, $\mathrm{diam}(A) \leq \delta$ and $\nu \in \mathbb{S}^{n-1}$, we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A \langle \nabla u, \omega\rangle^2 \ \mathrm{d}x + C(\delta)\int_{J_u\cap A} \frac{|\mathbf{M}(a)\nu|}{\langle \mathbf{M}(a)\nu, \nu\rangle^{\frac{1}{2}}}\langle \omega, \nu_u\rangle \ \mathrm{d}\mathcal{H}^{n-1},$$

where $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$.

We denote by A an arbitrary open subset of Ω such that $\mathrm{dam}(A) \leq \delta$ and we fix $a \in A$. Let $\nu \in \mathbb{S}^{n-1}$ be fixed. According to Hölder 1.2 and Ellipticity 1.1 conditions, we have

$$\begin{aligned} \forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad |\langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle - \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle| &\leq \theta\delta^\alpha |\mathbf{v}|^2, \\ &\leq \theta\delta^\alpha\lambda^{-1}\langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle. \end{aligned}$$

Then, we get

$$\forall (x, \mathbf{v}) \in A \times \mathbb{R}^n, \quad \langle \mathbf{M}(x)\mathbf{v}, \mathbf{v}\rangle \geq (1 - \theta\delta^\alpha\lambda^{-1})\langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle$$

We set $C(\delta) = 1 - \theta\delta^\alpha\lambda^{-1}$. Then, we may write

$$\forall x \in A, \quad \langle \mathbf{M}(x)\nabla z_k(x), \nabla z_k(x)\rangle^2 \geq C(\delta)\langle \mathbf{M}(a)\nabla z_k(x), \nabla z_k(x)\rangle^2.$$

As $\mathbf{M}(a)$ is a symmetric definite positive matrix, Cauchy-Schwartz inequality gives

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\nu, \nu\rangle\langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle \geq \langle \mathbf{M}(a)\nu, \mathbf{v}\rangle^2,$$

which is equivalent to

$$\forall \mathbf{v} \in \mathbb{R}^n, \quad \langle \mathbf{M}(a)\mathbf{v}, \mathbf{v}\rangle \geq \frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle}\left\langle \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}, \mathbf{v}\right\rangle^2. \tag{3.12}$$

We set $\omega = \frac{\mathbf{M}(a)\nu}{|\mathbf{M}(a)\nu|}$. If we apply inequality (3.12) to $F_{\varepsilon_k}(u_k, z_k; A)$, we have

$$F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A \left(|\nabla u_k|^2(1 - z_k^2)^2 + C(\delta)\frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle}\varepsilon_k\langle \omega, \nabla z_k\rangle^2 + \frac{z_k^2}{4\varepsilon_k}\right) \mathrm{d}x.$$

With the notation introduced in (1.5), $(v)_y$ is the function defined on A_ω^y as $(v)_y(t) = v(y + t\omega)$. According to Lemma 2.1, we have $\nabla(u_k)_y(t) = \langle \nabla u(y + t\omega), \omega\rangle$ and $\nabla(z_k)_y(t) = \langle \nabla z(y + t\omega), \omega\rangle$, so Fubini Theorem gives

$$F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{A_\omega}\int_{A_\omega^y} \left(|\nabla(u_k)_y|^2(1 - ((z_k)_y)^2)^2 + C(\delta)\frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle}\varepsilon_k|\nabla(z_k)_y|^2 + \frac{((z_k)_y)^2}{4\varepsilon_k}\right) \ \mathrm{d}t \ \mathrm{d}\mathcal{H}^{n-1}(y).$$

With the one-dimensional notations (3.5), it gives

$$F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{A_\omega} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A_\omega^y) \ \mathrm{d}\mathcal{H}^{n-1}(y),$$

where $m = C(\delta)\frac{|\mathbf{M}(a)\nu|^2}{\langle \mathbf{M}(a)\nu, \nu\rangle}$ for any $x \in A$. Fatou lemma yields

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{A_\omega} \liminf_{k\to\infty} G_{\varepsilon_k}((u_k)_y, (z_k)_y; A_\omega^y) \ \mathrm{d}\mathcal{H}^{n-1}(y)$$

and then

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{A_\omega} G_-((u)_y; A_\omega^y) \ \mathrm{d}\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A)$ is finite, we deduce that $G_-((u)_y; A_\omega^y)$ is finite for $\mathcal{H}^{n-1}$ almost every $y \in A_\omega$. We may apply Proposition 3.1 with $I = A_\omega^y$ and $u = (u)_y$, it gives that $(u)_\omega^y \in \mathrm{SBV}(A_\omega^y)$ for $\mathcal{H}^{n-1}$ almost every $y \in A_\omega$ and we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_{A_\omega} \left[\int_{A_\omega^y} |\nabla(u)_y|^2 \mathrm{d}t + \mathcal{H}^0(J_{(u)_y} \cap A_\omega^y) m^{\frac{1}{2}} \right] \mathrm{d}\mathcal{H}^{n-1}(y).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; A)$ is finite, Theorem 1.4 implies

$$\int_{A_\omega} \left[\int_{A_\omega^y} |\nabla(u)_y|^2 \mathrm{d}t + \mathcal{H}^0(J_{(u)_y} \cap A_\omega^y) m^{\frac{1}{2}} \right] \mathrm{d}\mathcal{H}^{n-1}(y) = \int_\Omega |\langle \nabla u, \omega \rangle|^2 \mathrm{d}x + \int_{J_u \cap A} m^{\frac{1}{2}} \langle \omega, \nu_u \rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

We deduce

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A |\langle \nabla u, \omega \rangle|^2 \mathrm{d}x + \int_{J_u \cap A} m^{\frac{1}{2}} \langle \omega, \nu_u \rangle \, \mathrm{d}\mathcal{H}^{n-1}.$$

If we replace m and ω by their value, it gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; A) \geq \int_A \langle \nabla u, \omega \rangle^2 \mathrm{d}x + C(\delta) \int_{J_u \cap A} \frac{\langle \mathbf{M}(a)\nu, \nu_u \rangle}{\langle \mathbf{M}(a)\nu, \nu \rangle^{\frac{1}{2}}} \, \mathrm{d}\mathcal{H}^{n-1}.$$

Second Step: We prove (3.11).

The function $x \to \frac{\nabla u(x)}{|\nabla u(x)|}$ is measurable in $U = \{x \in \Omega : \nabla u(x) \neq 0\}$. According to Lusin Theorem (1.45 of [6]), there exists an increasing sequence of compacts $(K_l)_l \subset U$ such that

$$\begin{cases} \mathcal{L}^n(U \setminus K_l) \leq \frac{1}{l}, \\ x \to \frac{\nabla u(x)}{|\nabla u(x)|} \text{ is continuous in } K_l. \end{cases}$$

Thus, for any $x \in K_l$, there exists $r > 0$ such that

$$y \in B(x, r) \quad \Rightarrow \left| \frac{\nabla u(x)}{|\nabla u(x)|} - \frac{\nabla u(y)}{|\nabla u(y)|} \right| \leq \frac{1}{l}, \tag{3.13}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [6]), there exists a countable, pairwise disjoint collection of balls $(B_i)_{i\in I}$ satisfying (3.13) such that

$$\forall i \in I, B_i \subset \Omega, \mathrm{diam}(B_i) \leq \delta, \qquad \mathcal{L}^n \left(K_l \setminus \bigcup_{i\in I} B_i \right) = 0.$$

For any $i \in I$, we fix $x_i \in B_i$ and we set $\nu_i = \frac{(\mathbf{M}(a))^{-1}\nabla u(x_i)}{|(\mathbf{M}(a))^{-1}\nabla u(x_i)|}$. According to *First Step*, with $A = B_i$, $a = x_i$ and $\nu = \nu_i$, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; B_i) \geq \int_{B_i} \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 \mathrm{d}x.$$

Moreover, we have

$$\begin{aligned} \forall x \in B_i \cap K_l, \quad \left| |\nabla u|^2 - \left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \right\rangle^2 \right| &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} \rangle^2 - \langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle^2 \right|, \\ &\leq \left| \langle \nabla u, \frac{\nabla u}{|\nabla u|} - \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle \langle \nabla u, \frac{\nabla u}{|\nabla u|} + \frac{\nabla u(x_i)}{|\nabla u(x_i)|} \rangle \right|, \\ &\leq \frac{2}{l} |\nabla u|^2. \end{aligned}$$

It gives

$$\int_{B_i\cap K_l}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x \geq \frac{l}{l+2}\int_{B_i\cap K_l}|\nabla u|^2\mathrm{d}x.$$

As $\liminf F_{\varepsilon_k}(u_k, z_k;\ \cdot\ )$ is supperaddditive (3.3) and non decreasing (3.4), we have

$$\begin{aligned}
\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k;\Omega) &\geq \sum_{i\in I}\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; B_i),\\
&\geq \sum_{i\in I}\int_{B_i}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x,\\
&\geq \sum_{i\in I}\int_{B_i\cap K_l}\left\langle \nabla u, \frac{\nabla u(x_i)}{|\nabla u(x_i)|}\right\rangle^2 \,\mathrm{d}x,\\
&\geq \frac{l}{l+2}\sum_{i\in I}\int_{B_i\cap K_l}|\nabla u|^2\,\mathrm{d}x,\\
&\geq \frac{l}{l+2}\int_{\cup_i B_i\cap K_l}|\nabla u|^2\,\mathrm{d}x,
\end{aligned}$$

As $\mathcal{L}^n(K_l\setminus \cup_i B_i)=0$, we deduce

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k;\Omega) \geq \frac{l}{l+2}\int_{K_l}|\nabla u|^2\,\mathrm{d}x$$

and taking the limit $l\to\infty$ gives

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k;\Omega) \geq \int_U |\nabla u|^2\,\mathrm{d}x = \int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

In particular, $\int_\Omega |\nabla u|^2\,\mathrm{d}x$ is finite. As u belongs to $\mathrm{SBV}(\Omega)$, according to Theorem 1.2, there exists a pairwise disjoint family $(C_i)_{i\in\mathbb{N}}$ of $\mathcal{C}^1$ compact manifolds and $M\in\Omega$ such that:

$$J_u = \mathcal{N}\cup\left(\bigcup_{i\in\mathbb{N}} C_i\right), \qquad \mathcal{H}^{n-1}(\mathcal{N})=0.$$

As $\liminf F_{\varepsilon_k}(u_k, z_k;\Omega)$ is finite, *First Step* and Theorem 1.4 imply that $\mathcal{H}^{n-1}(J_u)$ is also finite. According to Ellipticity condition 1.1, we deduce that $\int_{J_u}\langle \mathbf{M}\nu_u,\nu_u\rangle^{\frac12}\,\mathrm{d}\mathcal{H}^{n-1}$ is finite. Then, for a fixed $\delta>0$, there exists $N\in\mathbb{N}$ such that

$$\int_{J_u\setminus\bigcup_{i=1}^N C_i}\langle \mathbf{M}\nu_u,\nu_u\rangle^{\frac12}\,\mathrm{d}\mathcal{H}^{n-1}\leq\delta. \tag{3.14}$$

We set $K=\bigcup_{i=1}^N C_i$ and $K_\tau=\{x\in\Omega\colon \mathrm{dist}(x,K)<\tau\}$. As $\int_\Omega|\nabla u|^2\,\mathrm{d}x$ is finite, there exists $\tau>0$ such that

$$\int_{\Omega\setminus K_\tau}|\nabla u|^2\,\mathrm{d}x\leq\delta. \tag{3.15}$$

With the same arguments as before, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k;\Omega\setminus\overline{K_\tau}) \geq \int_{\Omega\setminus\overline{K_\tau}}|\nabla u|^2\,\mathrm{d}x. \tag{3.16}$$

As $x\to\frac{\mathbf{M}(x)\nu_u(x)}{\langle\mathbf{M}(x)\nu_u(x),\nu_u(x)\rangle^{\frac12}}$ is continuous in K, for any $x\in K$ there exists $r>0$ such that

$$y\in B(x,r)\cap K \quad\Rightarrow\left|\frac{\mathbf{M}(x)\nu_u(x)}{\langle\mathbf{M}(x)\nu_u(x),\nu_u(x)\rangle^{\frac12}}-\frac{\mathbf{M}(y)\nu_u(y)}{\langle\mathbf{M}(y)\nu_u(y),\nu_u(y)\rangle^{\frac12}}\right|\leq\delta. \tag{3.17}$$

As a consequence of Besicovitch Covering Theorem (2.18 of [6]), there exists a countable, pairwise disjoint collection of balls $(\widetilde{B}_j)_{j\in\tilde{I}}$ satisfying (3.17) such that

$$\forall j \in \tilde{I}, \widetilde{B}_j \subset K_\tau, \mathrm{diam}(\widetilde{B}_j) \leq \delta, \qquad \mathcal{H}^{n-1}\left(K \setminus \bigcup_{j\in\tilde{I}} \widetilde{B}_j\right) = 0.$$

For any $j \in \tilde{I}$, we fix $\tilde{x}_j \in \widetilde{B}_j$. According to *First Step*, with $A = \widetilde{B}_j$, $a = \tilde{x}_j$ and $\nu = \nu_u(\tilde{x}_j)$, we get

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \widetilde{B}_j) \geq C(\delta) \int_{J_u\cap\widetilde{B}_j} \frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{\frac{1}{2}}} \,\mathrm{d}\mathcal{H}^{n-1}.$$

For any $x \in \widetilde{B}_j \cap K$, we have

$$\begin{aligned} \left|\frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(x)\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{\frac{1}{2}}} - \langle \mathbf{M}(x)\nu_u(x), \nu_u(x)\rangle^{\frac{1}{2}}\right| &\leq \left|\frac{\mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j)}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{\frac{1}{2}}} - \frac{\mathbf{M}(x)\nu_u(x)}{\langle \mathbf{M}(x)\nu_u(x), \nu_u(x)\rangle^{\frac{1}{2}}}\right|, \\ &\leq \delta. \end{aligned}$$

It gives

$$\int_{\widetilde{B}_j\cap K} \frac{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u\rangle}{\langle \mathbf{M}(\tilde{x}_j)\nu_u(\tilde{x}_j), \nu_u(\tilde{x}_j)\rangle^{\frac{1}{2}}} \,\mathrm{d}\mathcal{H}^{n-1} \geq \int_{\widetilde{B}_j\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\widetilde{B}_j \cap K).$$

As $\liminf F_{\varepsilon_k}(u_k, z_k; \,\cdot\,)$ is supperaddditive (3.3) and non decreasing (3.4), we have

$$\begin{aligned} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; K_\tau) &\geq \sum_{j\in\tilde{I}} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \widetilde{B}_j), \\ &\geq \sum_{j\in\tilde{I}} C(\delta) \left(\int_{\widetilde{B}_j\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\widetilde{B}_j \cap K)\right), \\ &\geq C(\delta) \left(\int_{\cup_j\widetilde{B}_j\cap K} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(\cup_j\widetilde{B}_j \cap K)\right), \end{aligned}$$

As $\mathcal{L}^n(K \setminus \widetilde{B}_j) = 0$, we deduce

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; K_\tau) \geq C(\delta) \left(\int_K \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(K)\right). \tag{3.18}$$

According to (3.16) and (3.18), we deduce

$$\begin{aligned} \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \Omega) &\geq \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; K_\tau) + \liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \Omega \setminus \overline{K_\tau}), \\ &\geq C(\delta) \left(\int_K \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta\mathcal{H}^{n-1}(K)\right) + \int_{\Omega\setminus\overline{K_\tau}} |\nabla u|^2 \,\mathrm{d}x. \end{aligned}$$

According to (3.15) and (3.14), we have

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \Omega) \geq C(\delta) \left(\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} - \delta - \delta\mathcal{H}^{n-1}(K)\right) + \int_\Omega |\nabla u|^2 \,\mathrm{d}x - \delta.$$

Letting $\delta \to 0^+$ concludes the proof

$$\liminf_{k\to\infty} F_{\varepsilon_k}(u_k, z_k; \Omega) \geq \int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \,\mathrm{d}\mathcal{H}^{n-1} + \int_\Omega |\nabla u|^2 \,\mathrm{d}x.$$

$\square$

3.2 The inequality for the *higher* Γ-limit

In this section we prove the following *upper* inequality of Γ-convergence (Theorem 3.1, *ii*)).

Proposition 3.2. *For $u \in \mathbb{B}(\Omega)\cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere, $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k) \le F(u).$$

We first prove a weaker result, where $\int_{J_u} \langle \mathbf{M}\nu, \nu\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ is replaced by its approximation with Minkowski content. Then, with regularity results of section 1.5, we generalize this result.

3.2.1 Approximation with anisotropic Minkowski content

We prove the following result.

Proposition 3.3. *For $u \in SBV(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k) \le \int_\Omega |\nabla u|^2 dx + \mathcal{M}^\star_{\mathbf{M}}(J_u),$$

where $\mathcal{M}^\star_{\mathbf{M}}$ is defined in 1.6.

Proof. We may assume that $|\nabla u| \in \mathrm{L}^2(\Omega)$, otherwise $F(u) = +\infty$ and the result is obvious. For the same reason, we may assume that $\mathcal{M}^\star_{\mathbf{M}}(J_u) < +\infty$. If $u \in \mathrm{W}^{1,2}(\Omega)$, then $J_u = \emptyset$ and the stationary sequence $u_k = u$, $z_k = 0$ is a solution. In the other case, $J_u \neq \emptyset$ and $(1 - z_k^2)^2$ has to be infinitesimal near of J_u. For $\rho > 0$, we set

$$(J_u)_\rho = \{x \colon \mathrm{d}_\phi^{J_u}(x) < \rho\}.$$

We separate Ω in three parts:

$$(J_u)_{b_k}, \quad (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}, \quad \Omega \setminus (J_u)_{a_k+b_k}$$

with

$$\left\{ \begin{array}{rcl} a_k &=& -4\varepsilon_k \ln(\varepsilon_k), \\ b_k &=& \varepsilon_k^2. \end{array} \right. \tag{3.19}$$

Let $\Psi_k \in \mathcal{C}_0^\infty(\Omega)$ such that $\Psi_k = 1$ in $(J_u)_{\frac{b_k}{2}}$ and $\Psi_k = 0$ in $\Omega \setminus (J_u)_{b_k}$. We set $u_k = (1 - \Psi_k)u$ and then $u_k = u$ in $\Omega \setminus (J_u)_{b_k}$. As $(b_k)_k$ converges to 0 then u_k converges to u almost everywhere.

We set $z_k = 1$ in $(J_u)_{b_k}$ and $z_k = \varepsilon_k^2$ in $\Omega \setminus (J_u)_{a_k+b_k}$. In $(J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$ we adopt the following construction. We introduce

$$\theta_k(t) = \varepsilon_k^2 \exp\left(\frac{t}{2\varepsilon_k}\right);$$

and we set

$$\tilde{z}_k(t) = \left\{ \begin{array}{ll} 1 & \forall t \in [0; b_k], \\ \theta_k(a_k + b_k - t) & \forall t \in]b_k; a_k + b_k];, \\ \varepsilon_k^2 & \forall t \in]a_k + b_k; +\infty[. \end{array} \right. \tag{3.20}$$

This is a continuous and decreasing function defined on $[0; +\infty[$, moreover it satisfies

$$\forall t \in]b_k; a_k + b_k[, \quad \varepsilon_k(\tilde{z}_k'(t))^2 = \frac{(\tilde{z}_k(t))^2}{4\varepsilon_k}. \tag{3.21}$$

We set $z_k = \tilde{z}_k \circ \mathrm{d}_\phi^{J_u}$. As z_k is constant in $(J_u)_{b_k} \cup (\Omega \setminus (J_u)_{a_k+b_k})$, we have

$$\begin{aligned} F_{\varepsilon_k}(u_k, z_k) &= \int_{\Omega\setminus(J_u)_{a_k+b_k}} |\nabla u|^2(1-\varepsilon_k^4)^2 \mathrm{d}x + \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} |\nabla u|^2(1-z_k^2)^2 \mathrm{d}x \\ &\quad + \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} \left(\varepsilon_k\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle + \frac{z_k^2}{4\varepsilon_k}\right) \mathrm{d}x \\ &\quad + \frac{\varepsilon_k^3}{4}\mathcal{L}^n(\Omega\setminus(J_u)_{a_k+b_k}) + \frac{1}{4\varepsilon_k}\mathcal{L}^n((J_u)_{b_k}) \end{aligned} \tag{3.22}$$

As $|\nabla u| \in \mathrm{L}^2(\Omega)$ and $(a_k + b_k)_k$ converges to 0, the first term of (3.22) converges to $\int_\Omega |\nabla u|^2 \mathrm{d}x$. As $\|z_k\|_{L^\infty} \leq 1$, so the second term converges to 0. As Ω is a bounded domain, the fourth term converges to 0. As $\mathcal{M}^\star_{\mathbf{M}}(J_u) < +\infty$, there exists $(\omega_k)_k$ a sequence which converges to 0^+ such that

$$\mathcal{L}^n((J_u)_{b_k}) \leq 2b_k(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k) \tag{3.23}$$

and then the fifth term is lower than $\frac{1}{2}\varepsilon_k(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k)$. So, the fifth term converges to 0. To compute the limit of $(F_{\varepsilon_k}(u_k, z_k))_k$, it remains to study the convergence of

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} \left(\varepsilon_k\langle \mathbf{M}\nabla z_k, \nabla z_k\rangle + \frac{z_k^2}{4\varepsilon_k}\right) \mathrm{d}x.$$

Proposition 1.1 yields

$$\begin{aligned} \forall (x,y) \in \Omega^2, \quad |\mathrm{d}_\phi^{J_u}(x) - \mathrm{d}_\phi^{J_u}(y)| &\leq \mathrm{d}_\phi(x,y), \\ &\leq \lambda^{-\frac{1}{2}}|x-y|. \end{aligned}$$

So, $\mathrm{d}_\phi^{J_u}$ is Lipschitzian and Rademacher theorem ensures that $\mathrm{d}_\phi^{J_u}$ exists for almost every $x \in \Omega$, in the sense of the approximate differentiability 1.10. Thus, for almost every $x \in (J_u)_{a_k+b_k} \setminus (J_u)_{b_k}$, we have

$$\nabla z_k = \tilde{z}_k' \circ \mathrm{d}_\phi^{J_u} \; \nabla \mathrm{d}_\phi^{J_u}.$$

It gives

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} \left(\varepsilon_k(\tilde{z}_k' \circ \mathrm{d}_\phi^{J_u})^2\langle \mathbf{M}\nabla \mathrm{d}_\phi^{J_u}, \nabla \mathrm{d}_\phi^{J_u}\rangle + \frac{(\tilde{z}_k \circ \mathrm{d}_\phi^{J_u})^2}{4\varepsilon_k}\right) \mathrm{d}x.$$

According to Proposition 1.1, we have

$$\langle \mathbf{M}(x)\nabla \mathrm{d}_\phi^{J_u}(x), \nabla \mathrm{d}_\phi^{J_u}(x)\rangle = 1$$

for almost every x, so we may write

$$A_k(z_k) = \int_{(J_u)_{a_k+b_k}\setminus(J_u)_{b_k}} \left(\varepsilon_k(\tilde{z}_k' \circ \mathrm{d}_\phi^{J_u})^2 + \frac{(\tilde{z}_k \circ \mathrm{d}_\phi^{J_u})^2}{4\varepsilon_k}\right) \langle \mathbf{M}\nabla \mathrm{d}_\phi^{J_u}, \nabla \mathrm{d}_\phi^{J_u}\rangle^{\frac{1}{2}} \mathrm{d}x.$$

We may apply Proposition 1.2 with $\Phi = \phi$ and $p = \mathrm{d}_\phi^{J_u}$, it gives

$$A_k(z_k) = \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}_k'(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k}\right) \left[\int_\Omega \langle \mathbf{M}D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t}\rangle^{\frac{1}{2}}\right] \mathrm{d}t. \tag{3.24}$$

We set

$$\begin{aligned} \mathcal{H}_{\mathbf{M}}(t) &= \int_\Omega \langle \mathbf{M}D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t}\rangle^{\frac{1}{2}}, \\ \mathcal{A}_{\mathbf{M}}(s) &= \int_0^s \mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t. \end{aligned}$$

Applying another time Proposition 1.2 gives

$$\begin{aligned}
\mathcal{A}_{\mathbf{M}}(s_2) - \mathcal{A}_{\mathbf{M}}(s_1) &= \int_{s_1}^{s_2} \left[\int_{\Omega} \langle \mathbf{M} D\mathbf{1}_{(J_u)_t}, D\mathbf{1}_{(J_u)_t} \rangle^{\frac{1}{2}} \right] \mathrm{d}t, \\
&= \int_{(J_u)_{s_2} \setminus (J_u)_{s_1}} \langle \mathbf{M} \nabla \mathrm{d}_\phi^{J_u}, \nabla \mathrm{d}_\phi^{J_u} \rangle^{\frac{1}{2}} \mathrm{d}x, \\
&= \mathcal{L}^n((J_u)_{s_2} \setminus (J_u)_{s_1}).
\end{aligned}$$

So, $\mathcal{A}_{\mathbf{M}} \in \mathrm{W}^{1,1}_{\mathrm{loc}}(]0;+\infty[)$ and $\nabla \mathcal{A}_{\mathbf{M}} = \mathcal{H}_{\mathbf{M}}$ almost everywhere. Using equality (3.21) and then integrating by parts (3.24) gives

$$\begin{aligned}
A_k(z_k) &= \int_{b_k}^{a_k+b_k} \left(\varepsilon_k \tilde{z}_k'(t)^2 + \frac{\tilde{z}_k(t)^2}{4\varepsilon_k} \right) \mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t, \\
&= \int_{b_k}^{a_k+b_k} \frac{\tilde{z}_k(t)^2}{2\varepsilon_k} \mathcal{H}_{\mathbf{M}}(t)\mathrm{d}t, \\
&= \frac{(a_k+b_k)^2}{2\varepsilon_k} \mathcal{A}_{\mathbf{M}}(a_k+b_k) - \frac{b_k}{2\varepsilon_k} \mathcal{A}_{\mathbf{M}}(b_k) - \frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}_k'(t)\tilde{z}_k(t)\mathcal{A}_{\mathbf{M}}(t).
\end{aligned}$$

The first term obviously converges to 0. As for (3.23), we have

$$\mathcal{A}_{\mathbf{M}}(b_k) \leq 2b_k(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k)$$

and then the second term converges to 0 too. As $s \to \mathcal{A}_{\mathbf{M}}(s)$ is non decreasing, then

$$\forall t \in [b_k; a_k + b_k], \quad \mathcal{A}_{\mathbf{M}}(t) \leq 2t(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k)$$

For the last term, we apply another time this inequality, it gives

$$-\frac{1}{\varepsilon_k} \int_{b_k}^{a_k+b_k} \tilde{z}_k'(t)\tilde{z}_k(t)\mathcal{A}_{\mathbf{M}}(t)\mathrm{d}t \leq -\frac{(\mathcal{M}^\star_{\mathbf{M}}(J_u) + \omega_k)}{\varepsilon_k} \int_{b_k}^{a_k+b_k} 2t\tilde{z}_k'(t)\tilde{z}_k(t)\mathrm{d}t. \tag{3.25}$$

Integrating by parts yields

$$\int_{b_k}^{a_k+b_k} 2t\tilde{z}_k'(t)\tilde{z}_k(t)\mathrm{d}t = (a_k+b_k)\tilde{z}_k(a_k+b_k)^2 - b_k\tilde{z}_k(b_k)^2 - \int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2\mathrm{d}t. \tag{3.26}$$

According to the definitions of (a_k, b_k, z_k) (3.19) and (3.20), we have

$$(a_k+b_k)\tilde{z}_k(a_k+b_k)^2 - b_k\tilde{z}_k(b_k)^2 = o(\varepsilon_k) \tag{3.27}$$

and equation (3.21) gives

$$\begin{aligned}
\int_{b_k}^{a_k+b_k} \tilde{z}_k(t)^2\mathrm{d}t &= 2\varepsilon_k \int_{b_k}^{a_k+b_k} |\tilde{z}_k'(t)|\tilde{z}_k(t)\mathrm{d}t, \\
&= \varepsilon_k(1-\varepsilon_k^2).
\end{aligned} \tag{3.28}$$

From (3.25), (3.26), (3.27) and (3.28) we deduce that $\limsup_k A_k(z_k) \leq \mathcal{M}^\star_{\mathbf{M}}(J_u)$ and, according to the decomposition (3.22), we have

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k) \leq \int_\Omega |\nabla u|^2 + \mathcal{M}^\star_{\mathbf{M}}(J_u).$$

To conclude the proof, it suffices to notice that $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$.

□

3.2.2 Approximation in the general setting

The goal of this section is to replace $\mathcal{M}^\star_{\mathbf{M}}(J_u)$ by $\int_{J_u} \langle \mathbf{M}\nu_u, \nu_u\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}$ in Proposition 3.3.

Proposition 3.4. *For $u \in SBV(\Omega) \cap L^\infty(\Omega)$, there exists a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere $(u_k)_k$ is bounded in $L^\infty(\Omega)$ and*

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k) \le F(u).$$

To prove this result, we need to introduce the following.

Definition 3.1. *Let $\mathcal{F}(\Omega)$ be the set of functions $u \in SBV(\Omega)$ for which, if $F(u) < +\infty$, then there exists a sequence $(u_k)_k \subset SBV(\Omega) \cap L^\infty(\Omega)$ converging almost everywhere to u, $\lim_{k\to\infty} F(u_k) = F(u)$ and*

$$\forall k \in \mathbb{N}, \quad \mathcal{M}_{\mathbf{M}}(J_{u_k}) = \int_{J_{u_k}} \langle \mathbf{M}\nu_{u_k}, \nu_{u_k}\rangle d\mathcal{H}^{n-1}.$$

Proof. Assume $\mathcal{F}(\Omega) = \mathrm{SBV}(\Omega)$. According to Proposition 3.3, by a diagonal extraction we may exhibit a sequence $(u_k, z_k)_k \subset \mathcal{D}_n(\Omega)$ such that $(u_k, z_k)_k$ converges to $(u, 0)$ almost everywhere and

$$\limsup_{k\to\infty} F_{\varepsilon_k}(u_k, z_k) \le F(u).$$

So, to prove the *upper* inequality of Γ-convergence, it suffices to show that $\mathcal{F}(\Omega) = \mathrm{SBV}(\Omega)$. We divide the proof in three *Claims*.

Claim 1: If $u \in SBV(\Omega)$ and $(u_k)_k \subset SBV(\Omega)$ satisfy

i) $(u_k)_k \subset \mathcal{F}(\Omega)$,

ii) $\lim_{k\to\infty} F(u_k) = F(u)$ *and* $F(u) < \infty$,

iii) $(u_k)_k$ *converges to* u*almost everywhere,*

then $u \in \mathcal{F}(\Omega)$.

With a diagonal extraction process, we exhibit a sequence $(u_l)_l$ which satisfies Definition 3.1.

Claim 2: It suffices to prove that $SBV(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$.

For $u \in \mathrm{SBV}(\Omega)$ and $N > 0$, we denote by $\overline{u}^N$ the truncated function defined in (1.9). So, $(\overline{u}^N)_N$ converges to u almost everywhere for $N \to \infty$. Moreover, Theorem 1.3 gives

$$D\overline{u}^N = \mathbf{1}_{-N\le u\le N} \nabla u \mathcal{L}^n + ((\overline{u}^N)^+ - (\overline{u}^N)^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u.$$

and then we deduce $\lim_{N\to\infty} F(\overline{u}^N) = F(u)$. According to *Claim 1*, it suffices to prove that $\mathrm{SBV}(\Omega) \cap L^\infty(\Omega) \subset \mathcal{F}(\Omega)$.

Claim 3: Let $u \in SBV(\Omega) \cap L^\infty(\Omega)$, we have $u \in \mathcal{F}(\Omega)$.

We may assume that $F(u) < +\infty$, otherwise the result is ensured. Let us extend $\mathbf{M}$ and u in $\Omega' = \Omega \cup U$ as in Proposition 1.6, so we have

$$\mathcal{H}^{n-1}(J_u \cap \partial\Omega) = 0.$$

According to [1], Theorem 1.3, there exists $v_k \in \mathrm{SBV}(\Omega')$ a minimizer of the following functional:

$$E^{u,k}(v) = k \int_{\Omega'} (v-u)^2 \mathrm{d}x + \int_{\Omega'} |\nabla v|^2 \mathrm{d}x + \int_{J_v} \langle \mathbf{M}\nu_v, \nu_v\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

In particular, $E^{u,k}(v_k) \leq E^{u,k}(u)$ gives

$$\forall k \in \mathbb{N}, \quad k \int_{\Omega'} (v_k - u)^2 \mathrm{d}x \leq F(u)$$

and then $(v_k)_k$ converges to u almost everywhere. As $u \in L^\infty(\Omega)$, Corollary 1.1 gives

$$\mathcal{M}^\star_{\mathbf{M}}(J_{v_k}) = \int_{J_{v_k}} \langle \mathbf{M}\nu_{v_k}, \nu_{v_k} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}.$$

We introduce the sequence of positive Radon measures $(\mu_k)_k$ and μ defined by

$$\begin{aligned} \forall B \in \mathcal{B}(\Omega'), \quad \mu_k(B) &= \int_B |\nabla v_k|^2 \mathrm{d}x + \int_{J_{v_k} \cap B} \langle \mathbf{M}\nu_{v_k}, \nu_{v_k} \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}, \\ \mu(B) &= \int_B |\nabla u|^2 \mathrm{d}x + \int_{J_u \cap B} \langle \mathbf{M}\nu_u, \nu_u \rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1}. \end{aligned}$$

According to [1], F is lower semi-continuous in SBV, it gives

$$\forall A \subset \Omega' \text{ open }, \quad \liminf_{k \to \infty} \mu_k(A) \geq \mu(A).$$

The inequality

$$\limsup_{k \to \infty} \mu_k(\Omega') \leq \mu(\Omega')$$

follows by the definition of v_k. According to [6], Proposition 1.80, the measures $(\mu_k)_k$ weakly converge to μ. Thus, $(\mu_k(B))_k$ converges to $\mu(B)$ if $\mu(\partial B) = 0$, and then $(\mu_k(\Omega))_k$ converges to $\mu(\Omega)$, that is $(F(v_k))_k$ converges to $F(u)$. According to *Claim 1*, we deduce that $u \in \mathcal{F}(\Omega)$.

□

3.3 Proof of Theorem 1.6 *ii)*

Proof. According to Theorem 1.6 *i)*, for any $\varepsilon > 0$, there exists $(u_\varepsilon, z_\varepsilon)$ a minimizer of E_ε. According to (2.1), with $N \geq \|g\|_{L^\infty(\Omega)}$, we have

$$\mathcal{L}^n(\{x \in \Omega \colon |u_\varepsilon(x)| > N\}) > 0 \quad \Rightarrow \quad E_\varepsilon(\overline{u}_\varepsilon^N, z_\varepsilon) < E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

We deduce that $\|u_\varepsilon\|_{L^\infty(\Omega)} \leq N$ for any $\varepsilon > 0$. Denoting $\omega_\varepsilon = u_\varepsilon(1 - z_\varepsilon^2)$, we have

$$\nabla \omega_\varepsilon = \nabla u_\varepsilon (1 - z_\varepsilon^2) - 2 u_\varepsilon z_\varepsilon \nabla z_\varepsilon.$$

It yields

$$\int_\Omega |\nabla \omega_\varepsilon| \mathrm{d}x \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} \left(\int_\Omega |\nabla u_\varepsilon|^2 (1 - z_\varepsilon^2)^2 \mathrm{d}x \right)^{\frac{1}{2}} + 2N \int_\Omega |\nabla z_\varepsilon| z_\varepsilon \mathrm{d}x. \tag{3.29}$$

Applying the inequality $2ab \leq a^2 + b^2$ with $a = \frac{z_\varepsilon^2}{2\varepsilon^{\frac{1}{2}}}$ and $b = \varepsilon^{\frac{1}{2}} |\nabla z_\varepsilon|$ gives

$$\int_\Omega |\nabla z_\varepsilon| z_\varepsilon \mathrm{d}x \leq \int_\Omega \varepsilon |\nabla z_\varepsilon|^2 \mathrm{d}x + \int_\Omega \frac{z_\varepsilon^2}{4\varepsilon} \mathrm{d}x. \tag{3.30}$$

According to Ellipticity condition 1.1, we get

$$\int_\Omega \varepsilon |\nabla z_\varepsilon|^2 \mathrm{d}x \leq \frac{1}{\lambda} E_\varepsilon(u_\varepsilon, z_\varepsilon). \tag{3.31}$$

By (3.29), (3.30) and (3.31), we deduce

$$\int_\Omega |\nabla \omega_\varepsilon| \mathrm{d}x \leq \mathcal{L}^n(\Omega)^{\frac{1}{2}} \left(E_\varepsilon(u_\varepsilon, z_\varepsilon) \right)^{\frac{1}{2}} + \left(1 + \frac{1}{\lambda} \right) E_\varepsilon(u_\varepsilon, z_\varepsilon).$$

According to Proposition 3.4, as $E \not\equiv +\infty$, then $E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k})$ is a bounded sequence. So, $(\omega_{\varepsilon_k})_k$ is bounded in $\mathrm{BV}(\Omega)$ and there exists a subsequence, still denoted by $(\omega_{\varepsilon_k})_k$ which converges almost everywhere to $\omega \in \mathrm{BV}(\Omega)$. As $\int_\Omega z_k^2 \,\mathrm{d}x \leq \varepsilon_k E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k})$, then $(z_k)_k$ converges to 0 in $\mathrm{L}^2(\Omega)$ and there exists a subsequence, still denoted $(z_k)_k$, which converges almost everywhere to 0. As $\omega_{\varepsilon_k} = u_{\varepsilon_k}(1 - z_{\varepsilon_k}^2)$, then $(u_{\varepsilon_k})_k$ converges almost everywhere to $u \in \mathbb{B}(\Omega) \cap \mathrm{L}^\infty(\Omega)$.

In [1], we have proved that E admits a minimizer $v \in \mathrm{SBV}(\Omega)$ and $v \in \mathrm{L}^\infty(\Omega)$. According to Theorem 3.1, *ii*), there exists $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k \subset \mathcal{D}_n(\Omega)$ such that $(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k})_k$ converges to $(v, 0)$ almost everywhere and

$$\limsup_{k\to\infty} E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}) \leq E(v).$$

According to Theorem 3.1, *i*), we get

$$\liminf_{k\to\infty} E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}) \geq E(u).$$

As $(u_{\varepsilon_k}, z_{\varepsilon_k})$ is a minimizer of E_{ε_k}, we have

$$\forall k \in \mathbb{N}, \quad E_{\varepsilon_k}(v_{\varepsilon_k}, \tilde{z}_{\varepsilon_k}) \geq E_{\varepsilon_k}(u_{\varepsilon_k}, z_{\varepsilon_k}).$$

We conclude that $E(v) \geq E(u)$ and then u is also a minimizer of E.

□

4 Appendix

4.1 Proof of Corollary 1.1

Proof. We separate the proof in two *Claims*: first we prove that, up to a scaling, a minimizer is an almost quasi minimizer. Then, we prove that the given result is still true after a change of scale.

Claim 1: For $\beta > 0$ and f function defined in Ω we denote by f_β the function defined in $\beta\Omega$ by

$$\forall x \in \beta\Omega, \quad f_\beta(x) = f\left(\frac{x}{\beta}\right).$$

Then, there exists $\beta > 0$ such that $\widetilde{v}_\beta \in SBV(\beta\Omega)$ is an almost-quasi minimizer of a free discontinuity problem.

Let $\tilde{v} \in SBV(\Omega)$ be a minimizer of $E_{\mathbf{M}}^{\alpha,h}$ in $\mathrm{SBV}(\Omega)$. We introduce

$$\forall t \in \mathbb{R}, \quad \psi(t) = \begin{cases} -\|h\|_{L^\infty(\Omega)} & \text{if } t \leq -\|h\|_{L^\infty(\Omega)}, \\ t & \text{if } |t| \leq \|h\|_{L^\infty(\Omega)}, \\ \|h\|_{L^\infty(\Omega)} & \text{if } t \geq \|h\|_{L^\infty(\Omega)}. \end{cases}$$

According to the decomposition (1.5), for $u \in \mathrm{SBV}(\Omega)$, we have

$$D(\psi\circ u) = \mathbf{1}_{\{x\colon |u(x)| \leq \|h\|_{L^\infty}\}} \nabla u \mathcal{L}^n + ((\psi\circ u)^+ - (\psi\circ u)^-)\nu_u \mathcal{H}^{n-1} \llcorner J_u$$

and then

$$\mathcal{L}^n(\{x \in \Omega\colon |u(x)| > \|h\|_{L^\infty}\}) > 0 \quad \Rightarrow \quad E_{\mathbf{M}}^{\alpha,h}(\psi\circ u) < E_{\mathbf{M}}^{\alpha,h}(u).$$

We deduce that $|\tilde{v}(x)| \leq \|h\|_{L^\infty(\Omega)}$ for any $x \in \Omega$. By an homothetic change of variable, $\tilde{v}_\beta$ is a minimizer of the following rescaled problem

$$\left\{\alpha\beta^2 \int_{\beta\Omega} (v - h_\beta)^2 \mathrm{d}x + \int_{\beta\Omega} |\nabla v|^2 \mathrm{d}x + \beta \int_{J_v} \langle \mathbf{M}_\beta \nu_v, \nu_v\rangle^{\frac{1}{2}} \mathrm{d}\mathcal{H}^{n-1} \colon v \in \mathrm{SBV}(\beta\Omega)\right\}.$$

Chapitre 4

Expérimentations numériques

Dans ce chapitre, nous présentons rapidement des pistes pour l'expérimentation numérique des modèles proposés. Comme notre cadre de travail est le calcul des variations, les méthodes que nous envisageons consistent pour l'essentiel à minimiser une énergie. Le schéma général pour calculer ce minimum consiste d'abord à calculer la variation première de cette énergie (*l'équation d'Euler*), puis nous utilisons un schéma de point fixe équivalent à l'annulation de la variation première. Enfin, nous déterminons pour quelles valeurs des paramètres cette méthode converge. Ce faisant, nous mettons en parallèle les formulations continues et discrètes afin de mettre en avant les analogies.

Comme pour le travail théorique, la présentation suit un certain développement logique. Nous démarrons notre étude avec un modèle bien connu, ce qui nous permet de tirer des enseignements que nous réutiliserons dans la suite sur les modèles plus spécifiques.

4.1 Une première approche

Les résultats de cette section constituent le volet numérique de la première partie de la thèse. L'hypothèse simplificatrice sous laquelle notre raisonnement s'appuyait était la binarité de l'image à segmenter. Dans une angiographie, par effet de l'injection d'un produit rehausseur de contraste dans le sang, un seuillage de l'image à 60% de l'intensité maximale permet de conserver l'ensemble du réseau sanguin et d'obtenir ainsi une image binaire (Figure 4.1.1). C'est sur cette image que nous allons réaliser la segmentation.

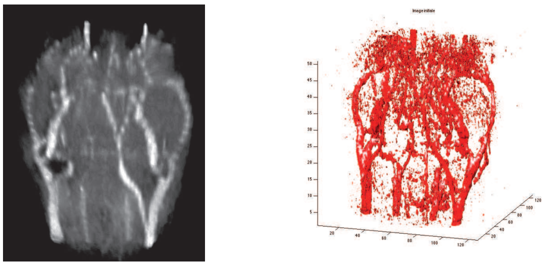

FIGURE 4.1.1 – Angiographie et seuillage à 60% de l'intensité maximale

L'énergie de Mumford-Shah pour un ensemble $A \subset \Omega$ est

$$\mathcal{E}(A) = \int_\Omega (\mathbf{1}_A - g)^2 \mathrm{d}x + \beta \mathcal{H}^{n-1}(\partial A).$$

Afin de calculer la variation première de cette énergie, nous en considérons plutôt une approximation. Dans [Mod87], il est démontré que, pour $\varepsilon > 0$, la fonctionnelle

$$E_\varepsilon(p) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_\Omega \left(9|\nabla p|^2 + \frac{p^2(1-p)^2}{\varepsilon} \right) \mathrm{d}x$$

admet un minimum $p_\varepsilon \in W^{1,2}(\Omega)$. Comme nous l'avons exposé dans l'article [BV14], le but est de calculer un minimiseur $\bar{p}$ de E_ε puis de déterminer une hauteur de seuillage de $\bar{p}$ qui permette d'effacer le bruit et de garder les tubes d'un certain rayon. Les équations que nous utilisons pour cela sont démontrées dans l'article [BV14] présenté p. 5

4.1.1 Minimisation de E_ε

En faisant l'analogie avec le cadre continu, nous allons introduire la méthode de minimisation de E_ε dans le cadre discret.

Cadre continu

Cherchons maintenant le moyen de calculer un minimiseur $\bar{p}$ de

$$\min\{E_\varepsilon(p) \colon p \in W^{1,2}(\Omega)\}.$$

Une condition nécessaire d'optimalité est

$$\nabla E_\varepsilon(\bar{p}) = 0.$$

Un calcul standard montre que $\bar{p}$ est une solution faible de l'équation aux dérivées partielles non linéaire suivante

$$\begin{cases} \bar{p} - g - 9\beta\varepsilon\Delta\bar{p} + \beta\dfrac{\bar{p}(1-\bar{p})(1-2\bar{p})}{\varepsilon} = 0 & \text{pour } x \in \Omega, \\ \dfrac{\partial \bar{p}}{\partial \mathbf{n}} = 0 & \text{pour } x \in \partial\Omega. \end{cases} \tag{4.1.1}$$

Cadre discret

La mise en oeuvre numérique se fait avec une discrétisation que nous allons préciser. La condition de Neumann est assurée par une réflexion de l'image par rapport à ses bords. Une image 3-D est un tableau $N \times N \times N$ que nous identifierons à une vecteur de l'espace euclidien $X = \mathbb{R}^{N^3}$ muni du produit scalaire usuel $\langle\ \cdot\ ,\ \cdot\ \rangle_X$. Dans le cadre de nos images obtenues par IRM, la discrétisation correspond à $N = 128$ et alors $N^3 \approx 10^6$. La grande dimension de ce système nous suggère de formuler le problème en différences finies explicites.

Pour cela, nous introduisons une version discrète de l'opérateur gradient. On pose $h = 1/N$ le pas spatial. Si $u \in X$, le gradient $\nabla_X u$ est un vecteur de $Y = X \times X \times X$ donné par

$$(\nabla_X u)_{i,j,k} = ((\nabla_X u)^1_{i,j,k}, (\nabla_X u)^2_{i,j,k}, (\nabla_X u)^3_{i,j,k}),$$

avec

$$\begin{aligned} (\nabla_X u)^1_{i,j,k} &= \begin{cases} \dfrac{u_{i+1,j,k} - u_{i,j,k}}{h} & \text{si } i < N, \\ 0 & \text{si } i = N, \end{cases} \\ (\nabla_X u)^2_{i,j,k} &= \begin{cases} \dfrac{u_{i,j+1,k} - u_{i,j,k}}{h} & \text{si } j < N, \\ 0 & \text{si } j = N, \end{cases} \\ (\nabla_X u)^3_{i,j,k} &= \begin{cases} \dfrac{u_{i,j,k+1} - u_{i,j,k}}{h} & \text{si } j < N, \\ 0 & \text{si } j = N. \end{cases} \end{aligned}$$

On introduit également une version discrète de l'opérateur de divergence défini par analogie avec le cadre continu en posant

$$\mathrm{div}_Y = -(\nabla_X)^*,$$

où $(\nabla_X)^*$ est l'opérateur adjoint de ∇_X, c'est-à-dire

$$\langle -\mathrm{div}_Y(U), u\rangle_X = \langle U, \nabla_X u\rangle_Y$$

pour tout $(U, u) \in Y \times X$, où $\langle \cdot , \cdot \rangle_Y$ est le produit scalaire usuel sur Y. On peut alors vérifier que la divergence discrète est donnée par la relation

$$(\mathrm{div}_Y(U))_{i,j,k} = \begin{cases} \dfrac{U^1_{i,j,k}}{h} & \text{si } i = 1 \\ \dfrac{U^1_{i,j,k} - U^1_{i-1,j,k}}{h} & \text{si } 1 < i < N \\ \dfrac{-U^1_{i-1,j,l}}{h} & \text{si } i = N \end{cases} + \quad \ldots$$

On utilisera aussi une version discrète du laplacien définie par

$$\Delta_X u = \mathrm{div}_Y(\nabla_X u).$$

On obtient alors

$$(\Delta_X u)_{i,j,k} = \frac{u_{i+1,j,k} + u_{i-1,j,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1} - 6u_{i,j,k}}{h^2}$$

en adoptant comme convention

$$\begin{cases} u_{0,j,k} = u_{1,j,k}, & u_{N+1,j,k} = u_{N,j,k}, \\ u_{i,0,k} = u_{i,1,k}, & u_{i,N+1,k} = u_{i,N,k}, \\ u_{i,j,0} = u_{i,j,1}, & u_{i,j,N+1} = u_{i,j,N} \end{cases}$$

pour tout (i, j, k) sur $\{1, \ldots, N\}^3$. On va remplacer le problème de minimisation de E_ε sur $W^{1,2}(\Omega)$ par la minimisation de

$$J_\varepsilon(u) \quad = \quad \langle u - g, u - g\rangle_X + \beta \left[9\varepsilon \langle \nabla_X u, \nabla_X u\rangle_Y + \frac{\langle u(1-u), u(1-u)\rangle_X}{\varepsilon} \right],$$

où $u \in X$ et $(u(1-u))_{i,j,k} = u_{i,j,k}(1 - u_{i,j,k})$. Sachant que J_ε est une fonction positive et polynomiale par rapport aux coordonnées de $u \in X$, elle admet donc un minimiseur. De plus, une condition nécessaire d'optimalité est $\nabla J_\varepsilon(\bar{u}) = 0_X$ (le gradient ∇ doit ici être entendu au sens classique d'une fonction dérivable $J_\varepsilon : X \to \mathbb{R}$). Un minimiseur $\bar{u} \in X$ vérifie donc

$$\bar{u} - g - 9\beta\varepsilon\Delta_X \bar{u} + \beta \frac{\bar{u}(1-\bar{u})(1-2\bar{u})}{\varepsilon} = 0_X.$$

Nous retrouvons dans cette équation l'analogue discret de l'équation (4.1.1). Dans cette équation les conditions aux bords de Neumann sont contenues dans la définition des opérateurs différentiels discrets. En posant $\delta > 0$, nous avons

$$\nabla J_\varepsilon(\bar{u}) = 0_X \Leftrightarrow \bar{u} - \delta\left[\bar{u} - g - 9\beta\varepsilon\Delta_X\bar{u} + \beta\frac{\bar{u}(1-\bar{u})(1-2\bar{u})}{\varepsilon}\right] = \bar{u} \quad (4.1.2)$$

et alors $\nabla J_\varepsilon(\bar{u}) = 0_X$ est équivalent à la recherche d'un point fixe pour la fonctionnelle

$$F_\delta(u) = u - \delta\left[u - g - 9\beta\varepsilon\Delta_X u + \beta\frac{u(1-u)(1-2u)}{\varepsilon}\right].$$

Nous introduisons alors l'algorithme suivant.

Algorithm 1 Algorithme de point fixe isotrope

Initialisation : $n = 0$; $u^0 = g$

Itération n : on pose

$$u^{n+1} = u^n - \delta\left[u^n - g - 9\beta\varepsilon\Delta_X u^n + \beta\frac{u^n(1-u^n)(1-2u^n)}{\varepsilon}\right].$$

Stop si un critère d'arrêt est satisfait.

Le résultat suivant assure la convergence de cet algorithme.

Théorème 4.1.1. *Si $g_{i,j,k} \in [0;1]$ pour tout $(i,j,k) \in \{1,\ldots,N\}^3$, $0 < \delta < 1$ et β vérifie*

$$\beta < \min\left\{\frac{\varepsilon}{324\varepsilon^2N^2+3}; \frac{\varepsilon}{108\varepsilon^2N^2+5.5}\right\}$$

alors l'algorithme de point fixe converge vers un point qui est l'unique minimiseur de J_ε.

Démonstration. Nous posons

$$\mathcal{K} = \left\{u \in X : \forall(i,j,k) \in \{1,\ldots,N\}^3,\ u_{i,j,k} \in \left[-\frac{1}{2}; \frac{3}{2}\right]\right\}.$$

Pour $u \in X$, nous posons

$$\tilde{u}_{i,j,k} = \begin{cases} 0 & \text{si } u_{i,j,k} < 0, \\ u_{i,j,k} & \text{si } 0 \leq u_{i,j,k} \leq 1, \\ 1 & \text{si } u_{i,j,k} > 1. \end{cases}$$

En constatant que $J_\varepsilon(\tilde{u}) \leq J_\varepsilon(u)$ et que l'inégalité est stricte s'il existe (i,j,k) tel que $u_{i,j,k} \notin [0;1]$, on en déduit que tout minimiseur de J_ε appartient à $\mathcal{K}$. Nous allons montrer que $F_\delta(\mathcal{K}) \subset \mathcal{K}$ et que F_δ est contractante sur $\mathcal{K}$. Soit $u \in \mathcal{K}$, on a alors

$$\forall (i,j,k) \in \{1,\ldots,N\}^3,\ u_{i,j,k} - \delta(u_{i,j,k} - g_{i,j,k}) \in \left[-\frac{1}{2} + \frac{\delta}{2}; \frac{3}{2} - \frac{\delta}{2}\right]. \quad (4.1.3)$$

Pour $(I,J,K) \in \{1,\ldots,N\}^3$, nous posons

$$\forall (i,j,k) \in \{1,\ldots,N\}^3, u_{i,j,k}^{I,J,K} = \cos\left(\frac{2\pi i I}{N}\right)\cos\left(\frac{2\pi j J}{N}\right)\cos\left(\frac{2\pi k K}{N}\right).$$

On peut vérifier que $u^{I,J,K}$ satisfait

$$\Delta_X u^{I,J,K} = \frac{2(\cos\left(\frac{2\pi I}{N}\right) + \cos\left(\frac{2\pi J}{N}\right) + \cos\left(\frac{2\pi K}{N}\right) - 3)}{h^2} u^{I,J,K}$$

et que $(u^{I,J,K})_{I,J,K}$ est une famille de N^3 vecteurs propres de X linéairement indépendants. Ils forment donc une base de vecteurs propres de Δ_X et en particulier nous avons

$$|\Delta_X u| \leq \frac{12}{h^2}|u| \quad (4.1.4)$$

pour tout $u \in X$. De plus, si $u \in \mathcal{K}$ alors

$$|u_{i,j,k}(1 - u_{i,j,k})(1 - 2u_{i,j,k})| \leq \frac{3}{2}. \quad (4.1.5)$$

De (4.1.3), (4.1.4) et (4.1.5) nous en déduisons que si $u \in \mathcal{K}$ alors

$$F_\delta(u) \in \left[-\frac{1}{2} + \frac{\delta}{2} - \frac{162\delta\beta\varepsilon}{h^2} - \frac{3\delta\beta}{2\varepsilon}; \frac{3}{2} - \frac{\delta}{2} + \frac{162\delta\beta\varepsilon}{h^2} + \frac{3\delta\beta}{2\varepsilon}\right].$$

Sachant que $0 < \delta < 1$, pour que $\mathcal{K}$ soit stable par F_k il suffit que

$$\frac{1}{2} - \frac{162\beta\varepsilon}{h^2} - \frac{3\beta}{2\varepsilon} > 0,$$

ce qui est équivalent à

$$\beta < \frac{\varepsilon}{324\varepsilon^2 N^2 + 3}.$$

Nous avons

$$|F_\delta(u) - F_\delta(v)| \leq (1-\delta)|u-v| + 9\delta\beta\varepsilon|\Delta_X(u-v)| + \frac{\beta\delta}{\varepsilon}|f(u) - f(v)|$$

où $f(u) = u(1-u)(1-2u)$. Sachant que $f'(t) \leq 11/2$ pour tout $t \in [-1/2; 3/2]$, d'après (4.1.4), nous avons

$$\begin{aligned} |F_\delta(u) - F_\delta(v)| &\leq (1-\delta)|u-v| + \frac{108\delta\beta\varepsilon}{h^2}|u-v| + \frac{11\beta\delta}{2\varepsilon}|u-v|, \\ &\leq \left(1 - \delta + \frac{108\delta\beta\varepsilon}{h^2} + \frac{11\beta\delta}{2\varepsilon}\right)|u-v|. \end{aligned}$$

Pour que F_δ soit contractante sur $\mathcal{K}$, il suffit donc que

$$\frac{108\beta\varepsilon}{h^2} + \frac{11\beta}{2\varepsilon} < 1$$

ce qui est équivalent à

$$\beta < \frac{\varepsilon}{108\varepsilon^2 N^2 + 5.5}.$$

Sachant que F_δ est contractante sur $\mathcal{K}$ et laisse stable $\mathcal{K}$, alors la suite définie par l'algorithme de point fixe est convergente vers l'unique point fixe de F_δ sur $\mathcal{K}$. Or, tout minimum u de J_ε vérifie $\nabla J_\varepsilon(u) = 0_X$, on en conclue, d'après l'équivalence (4.1.2), que tout minimum de J_ε est un point fixe de F_δ et le théorème est démontré.

□

4.1.2 Calcul de la hauteur de seuillage

Grâce à l'étude réalisée dans le premier article (Théorème 3.6 dans [BV14]), nous savons que le profil q d'une solution sur une tranche Σ_α (Figure 4.1.2) est donné par la solution de l'équation

$$\begin{cases} -9\beta\varepsilon\,(\omega\bar{q}')' + \omega f_{\beta,\varepsilon}(\bar{q}) = 0 \text{ sur } \left[-\frac{\alpha}{2}; \frac{\alpha}{2}\right], \\ \bar{q}(-\frac{\alpha}{2}) = \bar{q}(\frac{\alpha}{2}) = 0, \end{cases} \tag{4.1.6}$$

où $[-\alpha; \alpha]$ est le support du profil, $\omega(r) = \pi\ell|r| + 2\pi r^2$ et

$$f_{\beta,\varepsilon}(t) = \frac{\beta}{\varepsilon}(2t^3 - 3t^2) + \left(1 + \frac{\beta}{\varepsilon}\right) t - 1$$

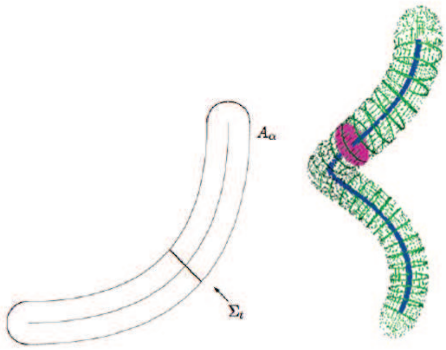

FIGURE 4.1.2 – Tranche Σ_α d'un tube A_α

Cette équation dépend de la longueur ℓ et du rayon α du tube. Nous distinguons alors les deux cas asymptotiques : lorsque le tube se réduit à une boule on a $\ell = 0$ et lorsque le rayon du tube est négligeable par rapport à sa longueur $\ell \to +\infty$. Nous en déduisons alors les deux équations suivantes.

$$\begin{cases} -9\beta\varepsilon r q'' - 9\beta\varepsilon q' + r f_{\beta,\varepsilon}(q) = 0 \\ \bar{q}(\frac{\alpha}{2}) = q(-\frac{\alpha}{2}) = 0 \end{cases}$$

$$\begin{cases} -9\beta\varepsilon r q'' - 18\beta\varepsilon q' + r f_{\beta,\varepsilon}(q) = 0 \\ \bar{q}(\frac{\alpha}{2}) = q(-\frac{\alpha}{2}) = 0 \end{cases}$$

À partir de ces deux équations, nous déterminons une hauteur de segmentation comprise entre les deux hauteurs maximales des solutions (Figure 4.1.3).

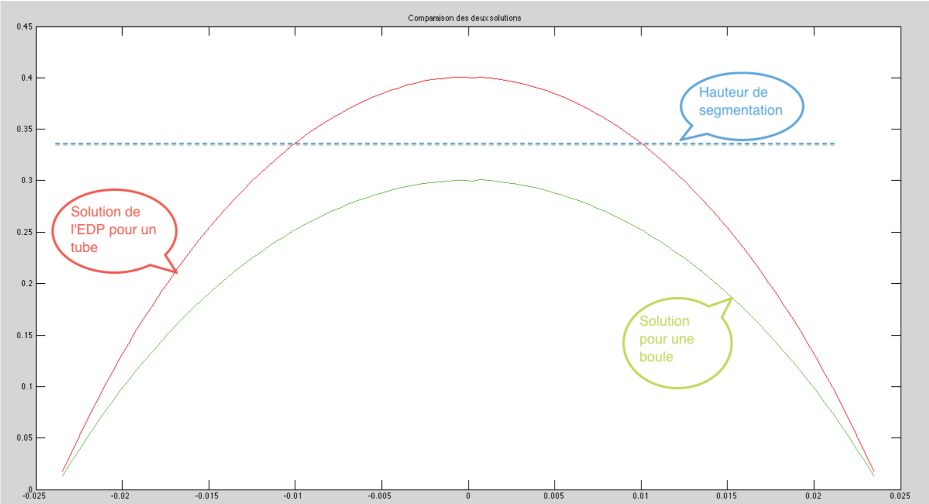

FIGURE 4.1.3 – Seuillage adapté à la segmentation

4.2 Un modèle anisotrope binaire

Nous présentons dans cette section la mise en œuvre numérique du modèle introduit dans [Vic15a] présenté p. 34. L'énergie à minimiser est donnée par

$$E_\varepsilon(p, \mathbf{M}) = \int_\Omega (p-g)^2 \mathrm{d}x + \beta \int_\Omega \left[9\varepsilon \langle \mathbf{M}\nabla p, \nabla p \rangle + \frac{p^2(1-p)^2}{\varepsilon} \right] \mathrm{d}x + \gamma \|\mathbf{M}\|_{W^{1,r}(\Omega)}.$$

Sachant que la dimension de l'espace des matrices symétriques est égale à 6, dans le cadre discret, un champ de matrice sur X sera défini par un vecteur de $Z = X^6$. Les 6 coefficients représentent les éléments sur-diagonaux de $\mathbf{M}$. Par abus de notation, pour $\mathbf{M} \in Z$, nous noterons

$$\mathbf{M}_{i,j,k} = \begin{pmatrix} (\mathbf{M}_{i,j,k})_1 & (\mathbf{M}_{i,j,k})_2 & (\mathbf{M}_{i,j,k})_3 \\ (\mathbf{M}_{i,j,k})_2 & (\mathbf{M}_{i,j,k})_4 & (\mathbf{M}_{i,j,k})_5 \\ (\mathbf{M}_{i,j,k})_3 & (\mathbf{M}_{i,j,k})_5 & (\mathbf{M}_{i,j,k})_6 \end{pmatrix}$$

pour tout $(i, j, k) \in \{1, \ldots, N\}^3$. De même, pour $U \in Y$, nous noterons

$$(\mathbf{M}U)_{i,j,k} = \begin{pmatrix} (\mathbf{M}_{i,j,k})_1 U^1_{i,j,k} + (\mathbf{M}_{i,j,k})_2 U^2_{i,j,k} + (\mathbf{M}_{i,j,k})_3 U^3_{i,j,k} \\ (\mathbf{M}_{i,j,k})_2 U^1_{i,j,k} + (\mathbf{M}_{i,j,k})_4 U^2_{i,j,k} + (\mathbf{M}_{i,j,k})_5 U^3_{i,j,k} \\ (\mathbf{M}_{i,j,k})_3 U^1_{i,j,k} + (\mathbf{M}_{i,j,k})_5 U^2_{i,j,k} + (\mathbf{M}_{i,j,k})_6 U^3_{i,j,k} \end{pmatrix}$$

pour tout $(i, j, k) \in \{1, \ldots, N\}^3$.

4.2.1 Schéma de minimisation de $\mathbf{M} \mapsto E_\varepsilon(p, \mathbf{M})$

Dans [Vic15a], nous avons proposé de prendre $\mathbf{M}$ sous la forme

$$\mathbf{M}(x) = \mathrm{Id}_n + \mu \mathbf{c}(x)(\mathbf{c}(x))^t$$

avec $\mathbf{c}(x) \in \mathbb{S}^{n-1}$, pour tout $x \in \Omega$. De plus, ce champ doit minimiser la somme de son action sur les tubes T et du terme régularisation (Figure 4.2.1)

$$G_p(\mathbf{c}) = \mathrm{Action}(p, \mathbf{c}) + \mathrm{Reg}(\mathbf{c}) = 9\beta\varepsilon\mu \int_\Omega \langle \mathbf{c}, \nabla p \rangle^2 \mathrm{d}x + \gamma \int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x,$$

où $r > n$ assure la continuité de $\mathbf{c}$.

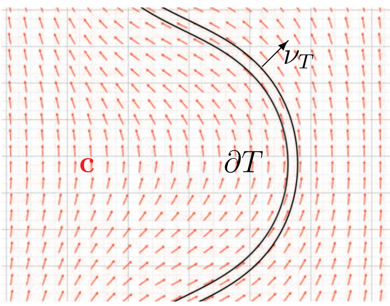

FIGURE 4.2.1 – Un champ $\mathbf{c}$ tangent à T

Le problème revient donc à minimiser G_p dans l'ensemble des champs de vecteurs unitaires de Ω. Dans le cadre discret, $\mathbf{c}$ appartient à Y et vérifie $|\mathbf{c}_{i,j,k}| = 1$ pour tout $(i, j, k) \in \{1, \ldots, N\}^3$. Pour $u \in X$ fixé, nous définissons l'analogue discret de G_p par

$$G_u(\mathbf{c}) = 9\beta\varepsilon\mu \langle \mathbf{c}, \nabla_X u \rangle_Y^2 + \gamma \left[\sum_{i=1}^{3} \langle \nabla_X \mathbf{c}^i, \nabla_X \mathbf{c}^i \rangle_Y^2 \right]^{r/2}.$$

Méthode de descente de gradient projeté

Le gradient de G_u, vu comme une fonction définie sur l'espace Y tout entier, est égal à

$$\begin{aligned}\nabla G_u(\mathbf{c}) &= 18\beta\varepsilon\mu\langle \mathbf{c}, \nabla_X u\rangle_Y \nabla_X u \\ &\quad -4\gamma \left[\sum_{i=1}^{3} \langle \nabla_X \mathbf{c}^i, \nabla_X \mathbf{c}^i\rangle_Y^2\right]^{(r-2)/2} \begin{pmatrix} \langle \nabla_X \mathbf{c}^1, \nabla_X \mathbf{c}^1\rangle_Y \Delta_X \mathbf{c}^1 \\ \langle \nabla_X \mathbf{c}^2, \nabla_X \mathbf{c}^2\rangle_Y \Delta_X \mathbf{c}^2 \\ \langle \nabla_X \mathbf{c}^3, \nabla_X \mathbf{c}^3\rangle_Y \Delta_X \mathbf{c}^3 \end{pmatrix}\end{aligned}$$

Nous introduisons alors l'algorithme de gradient projeté suivant.

Algorithm 2 Algorithme de gradient projeté

Initialisation : $n = 0$; *description dans la section suivante.*

Itération n : on pose

$$\mathbf{d}^{n+1} = \mathbf{c}^n - \delta\nabla G_u(\mathbf{c}^n),$$

$$\mathbf{c}^{n+1}_{i,j,k} = \frac{\mathbf{d}^n_{i,j,k}}{|\mathbf{d}^n_{i,j,k}|_{\mathbb{R}^3}}$$

pour tout $(i, j, k) \in \{1, \dots, N\}^3$.

Stop si un critère d'arrêt est satisfait.

Initialisation de l'algorithme 2

Dans le cas de la dimension $n = 3$ où les calculs sont forcément plus importants, nous proposons une initialisation possible de l'algorithme 2. Notre choix cherche à initialiser le champ $\boldsymbol{c}$ dans la *direction des tubes*. Son principe réside dans le fait que sur une surface lisse on peut trouver en tout point (non dégénéré) deux lignes de courbure principales. Pour une surface tubulaire la direction de plus faible courbure est dirigée dans la direction du tube (en rouge sur la Figure 4.2.2).

En reprenant la modélisation introduite dans la section 1.1 de l'introduction, nous modélisons le bord d'un tube comme une ligne de niveau d'une fonction. Pour une surface $S \subset \mathbb{R}^3$ donnée de manière implicite par

$$S := \{x \in \mathbb{R}^3 \mid g(x) = m\},$$

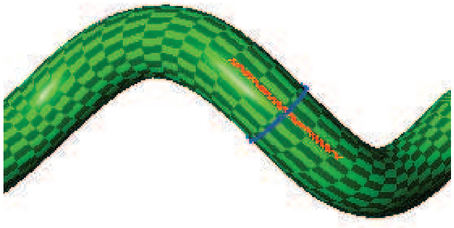

FIGURE 4.2.2 – Les deux lignes de courbure principales

nous considérons l'application définie par $\Psi(x) = \frac{\nabla g(x)}{|\nabla g(x)|}$. La différentielle de Ψ est alors

$$\begin{aligned} D\Psi(x) &= \frac{1}{|\nabla g(x)|}\left(Hg(x) - \Psi(x)(\Psi(x))^t\right), \\ &= \frac{1}{|\nabla g(x)|}\left(Hg(x) - P_{\Psi(x)}\right), \\ &= \frac{P_{\Psi(x)^\perp} \circ Hg(x)}{|\nabla g(x)|} \end{aligned}$$

où $Hg(x)$ est la hessienne de g en x, $P_{\Psi(x)}$ et $P_{\Psi(x)^\perp}$ sont les projections orthogonales sur $\mathrm{Vect}(\Psi(x))$ et sur son orthogonal. La restriction de la différentielle de Ψ au plan tangent à la surface $\Psi(x)^\perp$ correspond à l'endomorphisme de Weingarten de la surface. Pour connaître les directions principales de la surface, il suffit de calculer les directions propres de cet endomorphisme. L'initialisation du champ $\mathbf{c}$ que nous proposons revient donc à effectuer les opérations suivantes.

1. Calculer $\nabla_X g$ et $H_X g$ la Hessienne discrète de g.
2. Déterminer $(V, W) \in Y^2$ tels que $\left(V_{i,j,k}, W_{i,j,k}, \frac{(\nabla_X g)_{i,j,k}}{|(\nabla_X g)_{i,j,k}|}\right)$ soit une base orthonormale de $\mathbb{R}^3$ pour tout $(i,j,k) \in \{1, \dots, N\}^3$.
3. Déterminer les vecteurs propres unitaires (x_1, x_2) et (y_1, y_2) de la matrice

$$\begin{pmatrix} \langle (H_X g)_{i,j,k} V_{i,j,k}, V_{i,j,k} \rangle_{\mathbb{R}^3} & \langle (H_X g)_{i,j,k} V_{i,j,k}, W_{i,j,k} \rangle_{\mathbb{R}^3} \\ \langle (H_X g)_{i,j,k} W_{i,j,k}, V_{i,j,k} \rangle_{\mathbb{R}^3} & \langle (H_X g)_{i,j,k} W_{i,j,k}, W_{i,j,k} \rangle_{\mathbb{R}^3} \end{pmatrix},$$

où les valeurs propres associées sont $\lambda^1_{i,j,k}$ et $\lambda^2_{i,j,k}$, avec $\lambda^1_{i,j,k} \leq \lambda^2_{i,j,k}$, pour tout $(i,j,k) \in \{1,\dots,N\}^3$.

4. Poser
$$\mathbf{c}_{i,j,k} = x_1 V_{i,j,k} + x_2 W_{i,j,k}.$$

Nous remarquons enfin que pour toutes les opérations précédentes nous pouvons donner une formule explicite.

4.2.2 Schéma de minimisation de $p \mapsto E_\varepsilon(p, \mathbf{M})$

Dans un premier temps, nous considérons que $\mathbf{M}$ est fixé et nous cherchons un moyen de calculer un minimiseur $\bar{p}$ de $\min\{E_\varepsilon(p,\mathbf{M})\colon p \in W^{1,2}(\Omega)\}$. Ainsi, $\bar{p}$ est une solution faible de l'équation aux dérivées partielles non linéaire suivante

$$\left\{\begin{array}{rcll} \bar{p} - g - 9\beta\varepsilon \mathrm{div}(\mathbf{M}\nabla\bar{p}) + \beta\dfrac{\bar{p}(1-\bar{p})(1-2\bar{p})}{\varepsilon} & = & 0 & \text{pour } x \in \Omega, \\ \dfrac{\partial \bar{p}}{\partial \mathbf{n}} & = & 0 & \text{pour } x \in \partial\Omega. \end{array}\right. \tag{4.2.1}$$

Dans le cadre discret, l'analogue du problème de minimisation de $p \mapsto E_\varepsilon(p,\mathbf{M})$ sur $W^{1,2}(\Omega)$ consiste en la minimisation de l'énergie suivante

$$J_{\varepsilon,\mathbf{M}}(u) \quad = \quad \langle u-g, u-g\rangle_X + \beta\left[9\varepsilon\langle \mathbf{M}\nabla_X u, \nabla_X u\rangle_Y + \frac{\langle u(1-u), u(1-u)\rangle_X}{\varepsilon}\right],$$

où $u \in X$. Sachant que J_ε est une fonction positive et polynomiale par rapport aux coordonnées de $u \in X$, elle admet donc un minimiseur. De plus, une condition nécessaire d'optimalité est $\nabla J_{\varepsilon,\mathbf{M}}(\bar{u}) = 0_X$. Un minimiseur $\bar{u} \in X$ vérifie donc

$$\bar{u} - g - 9\beta\varepsilon \mathrm{div}_Y(\mathbf{M}\nabla_X \bar{u}) + \beta\frac{\bar{u}(1-\bar{u})(1-2\bar{u})}{\varepsilon} = 0_X.$$

Nous retrouvons dans cette équation l'analogue discret de l'équation (4.2.1). En posant $\delta > 0$, nous avons

$$\nabla J_{\varepsilon,\mathbf{M}}(\bar{u}) = 0_X \Leftrightarrow \bar{u} - \delta\left[\bar{u} - g - 9\beta\varepsilon \mathrm{div}_Y(\mathbf{M}\nabla_X \bar{u}) + \beta\frac{\bar{u}(1-\bar{u})(1-2\bar{u})}{\varepsilon}\right] = \bar{u}$$

et alors $\nabla J_{\varepsilon,\mathbf{M}}(\bar{u}) = 0_X$ est équivalent à la recherche d'un point fixe pour la fonctionnelle

$$F_{\delta,\mathbf{M}}(u) = u - \delta\left[u - g - 9\beta\varepsilon \mathrm{div}_Y(\mathbf{M}\nabla_X u) + \beta\frac{u(1-u)(1-2u)}{\varepsilon}\right].$$

Nous introduisons alors l'algorithme suivant.

Algorithm 3 Algorithme de point fixe anisotrope

Initialisation : $n = 0$; $u^0 = g$

Itération n : on pose

$$u^{n+1} = u^n - \delta\left[u^n - g - 9\beta\varepsilon \mathrm{div}_Y(\mathbf{M}\nabla_X u^n) + \beta\frac{u^n(1-u^n)(1-2u^n)}{\varepsilon}\right].$$

Stop si un critère d'arrêt est satisfait.

Le résultat suivant assure la convergence de cet algorithme.

Théorème 4.2.1. *Soit $\mathbf{M} \in Z$ et $\Lambda > 0$ tels que*

$$\langle \mathbf{M}_{i,j,k} U, U\rangle_{\mathbf{R}^3} \leq \Lambda |U|^2_{\mathbf{R}^3}$$

pour tout pour tout $(i,j,k) \in \{1,\ldots,N\}^3$ et pour tout $U \in \mathbb{R}^3$. Soit $g \in X$ tel que $g_{i,j,k} \in [0;1]$ pour tout $(i,j,k) \in \{1,\ldots,N\}^3$, $0 < \delta < 1$ et β tel que

$$\beta < \min\left\{\frac{\varepsilon}{324\Lambda\varepsilon^2 N^2 + 3}; \frac{\varepsilon}{108\Lambda\varepsilon^2 N^2 + 5.5}\right\}$$

alors l'algorithme 3 converge vers un point qui est l'unique minimiseur de $J_{\varepsilon,\mathbf{M}}$.

Remarque 4.2.1. *La condition sur $\mathbf{M}$ est une conséquence directe de la condition d'ellipticité introduite dans le deuxième chapitre (voir [Vic15a]).*

Démonstration. Comme pour la preuve du Théorème 4.1.1, tout minimum de $J_{\varepsilon,\mathbf{M}}$ appartient à l'ensemble

$$\mathcal{K} = \left\{u \in X : \forall (i,j,k) \in \{1,\ldots,N\}^3,\ u_{i,j,k} \in \left[-\frac{1}{2}; \frac{3}{2}\right]\right\}.$$

Nous allons montrer que $F_{\delta,\mathbf{M}}(\mathcal{K}) \subset \mathcal{K}$ et que $F_{\delta,\mathbf{M}}$ est contractante sur $\mathcal{K}$. D'après (4.1.4), nous avons

$$|\Delta_X u| \leq \frac{12}{h^2}|u|$$

pour tout $u \in X$. Les fonctions

$$\begin{array}{rcl} \Delta_X : X & \to & \mathbb{R} \\ u & \mapsto & \Delta_X u \end{array} \qquad \begin{array}{rcl} \Delta_{\mathbf{M}} : X & \to & \mathbb{R} \\ u & \mapsto & \mathrm{div}_Y(\mathbf{M}\nabla_X u) \end{array},$$

étant symétriques pour le produit scalaire $\langle \cdot\, , \, \cdot \rangle_X$, on a alors

$$\begin{array}{rcl} \|\Delta_{\mathbf{M}}\| & = & \sup\{\langle \mathrm{div}_Y(\mathbf{M}\nabla_X u), u\rangle_X : \quad u \in X, |u|_X = 1\}, \\ & = & \sup\{\langle (\mathbf{M}\nabla_X u), \nabla_X u\rangle_Y : \quad u \in X, |u|_X = 1\}, \\ & \leq & \Lambda \sup\{\langle \nabla_X u, \nabla_X u\rangle_Y : \quad u \in X, |u|_X = 1\}, \\ & \leq & \Lambda \sup\{\langle \Delta_X u, u\rangle_X : \quad u \in X, |u|_X = 1\}, \\ & \leq & \Lambda \|\Delta_X\|. \end{array}$$

Nous en déduisons

$$|\Delta_{\mathbf{M}} u| \leq \frac{12\Lambda}{h^2}|u|$$

pour tout $u \in X$. En reprenant les arguments de la preuve du Théorème 4.1.1, nous montrons que pour que $F_{\delta,\mathbf{M}}$ soit contractante et laisse stable $\mathcal{K}$, il suffit que

$$\beta < \min\left\{\frac{\varepsilon}{324\Lambda\varepsilon^2 N^2 + 3}; \frac{\varepsilon}{108\Lambda\varepsilon^2 N^2 + 5.5}\right\}.$$

et alors l'algorithme 3 converge vers un point qui est l'unique minimiseur de $J_{\varepsilon,\mathbf{M}}$. $\square$

4.3 Interprétation géométrique des paramètres du modèle

L'étude qui suit nous fournit des informations pour fixer les paramètres du modèle.

4.3.1 La largeur de la phase ε

Dans la partie théorique de la thèse, pour la démonstration de l'inégalité supérieure de Γ-convergence, nous avons démontré qu'un candidat optimal de profil pour la transition entre $\{x\colon z(x) = 0\}$ et $\{x\colon z(x) = 1\}$ est de *largeur* $\varepsilon|\ln(\varepsilon)|$ (relations (5.5) dans [Vic15a] et (3.19) dans [Vic15d]). Afin de représenter numériquement cette transition de phase dans le cas discret, la largeur de cette transition de phase doit au moins être égale au pas spatial. D'autre part, l'approximation par Γ-convergence du modèle suppose que ε soit petit, nous fixons donc

$$\varepsilon = h = \frac{1}{N}.$$

4.3.2 Le paramètre d'élongation μ

Nous avons montré dans [Vic15b] que μ correspond à l'élongation de la boule unité pour la métrique

$$\phi(x, \mathbf{v}) = \langle \mathbf{M}^{-1}\mathbf{v}, \mathbf{v}\rangle^{1/2}$$

qui est la métrique associée à la mesure de la surface $\int_S \langle \mathbf{M}\nu, \nu\rangle^{1/2} \mathrm{d}\mathcal{H}^{n-1}$. en définissant $\mathbf{M} = \mathrm{Id}_n + \mu\mathbf{c}\mathbf{c}^t$, la boule unité pour cette métrique est un ellipsoïde allongé dans la direction de $\mathbf{c}(x)$ d'un rapport $\sqrt{1+\mu}$ (Figure 4.3.1).

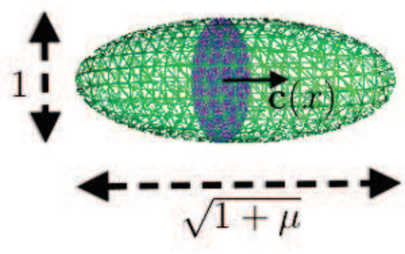

FIGURE 4.3.1 – Boule unité associée à ϕ

4.3.3 Le paramètre de courbure γ

Le paramètre γ pondère le terme $\int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x$ dans l'énergie associée au calcul de $\mathbf{M} = \mathrm{Id}_n + \mu\mathbf{c}\mathbf{c}^t$. Il pénalise donc les variations de $\mathbf{c}$. Soit $\phi_t(x)$ le flot associé au champ $\mathbf{c}$. Là où c'est défini, nous avons alors

$$\frac{\mathrm{d}\phi_t}{\mathrm{d}t}(x) = \mathbf{c}(\phi_t(x)).$$

Comme $\mathbf{c}$ est un champ de vecteurs unitaires et tangents au tube, alors $|\mathrm{d}^2\phi_t/\mathrm{d}t^2(x)|$ est égal à la courbure du tube en x. Nous en déduisons que $|D\mathbf{c}(x)\cdot\mathbf{c}(x)|$ est égal à la courbure du tube en x. Sachant que

$$|D\mathbf{c}(x)\cdot\mathbf{c}(x)| \leq \|D\mathbf{c}(x)\|,$$

alors $\int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x$ domine la norme de Sobolev $\mathrm{W}^{1,r}$ de la fonction qui associe à chaque point x la courbure du flot associé au champ $\mathbf{c}$.

En général, $\int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x$ domine strictement la norme de Sobolev de la courbure associé au flot. Par exemple, dans la Figure 4.3.2, il est clair que $\int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x > 0$ mais $|D\mathbf{c}(x)\cdot\mathbf{c}(x)| = 0$ pour tout $x \in \Omega$.

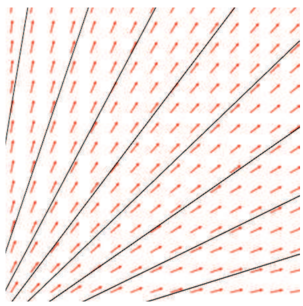

FIGURE 4.3.2 – Flot sans courbure associé à un champ non constant

Le terme $\int_\Omega \|D\mathbf{c}\|^r \mathrm{d}x$ contrôle donc la courbure des lignes de flot mais aussi la variation de l'orientation des lignes de flot entre elles.

4.3.4 Le paramètre de régularisation β

Pour la convergence des algorithmes, les théorèmes 4.1.1 et 4.2.1 nous donnent

$$\beta < \min\left\{\frac{\varepsilon}{324\Lambda\varepsilon^2N^2+3}; \frac{\varepsilon}{108\Lambda\varepsilon^2N^2+5.5}\right\}$$

avec Λ le paramètre d'ellipticité. dans le cas où nous posons $\mathbf{M} = \mathrm{Id}_n + \mu\mathbf{c}\mathbf{c}^t$, il est clair que $\Lambda = 1+\mu$. D'autre part, nous avons posé $\varepsilon = 1/N$, la condition de convergence de l'algorithme 4.2.1 se réduit alors à

$$\beta < \frac{1}{N(324(1+\mu)+3)}.$$

4.4 Exemples

Nous considérons tout d'abord une image binaire 2D constituée d'un tube et d'une boule (Figure 4.4.1). La boule a un diamètre égal à la section du tube, c'est-à-dire 4 pixels. Comparons sur cette image l'effet des algorithmes isotrope 1 et anisotrope 3.

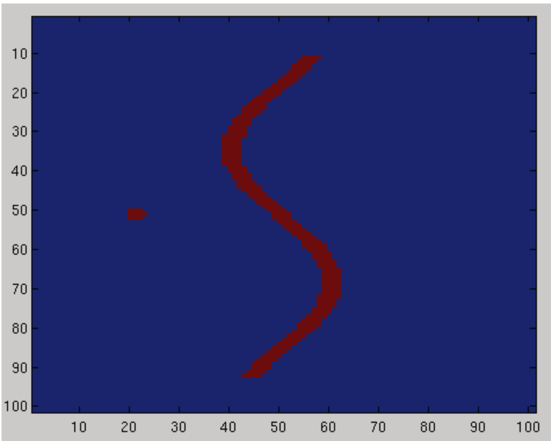

FIGURE 4.4.1 – Image initiale

La Figure 4.4.2 est le résultat de l'algorithme isotrope 1 après 100 itérations.

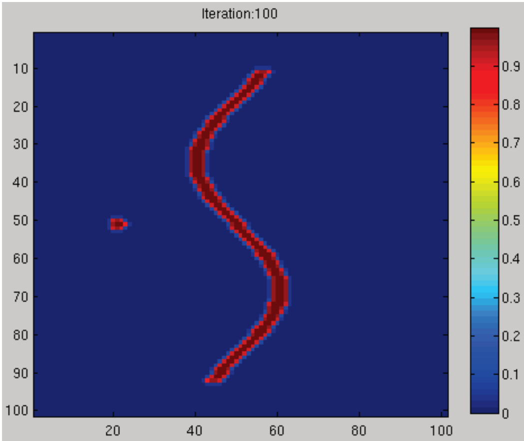

FIGURE 4.4.2 – Cas isotrope

Avec une anisotropie correspondant à $\mu = 20$, nous calculons le champ **c** par l'algorithme 2.

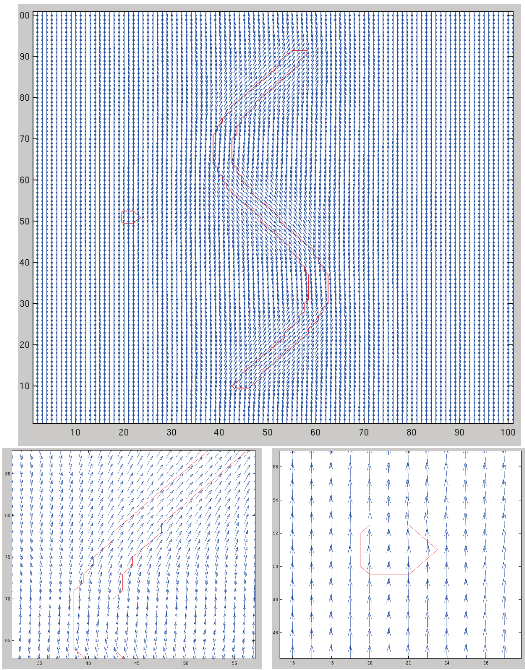

FIGURE 4.4.3 – Champ **c**

Comme nous le souhaitions, alors que le champ obtenu est tangent au tube, il ne l'est pas pour la boule (voir Figure 4.4.3). En conservant le même jeu de paramètres que dans le cas isotrope, nous obtenons par l'algorithme 3 l'image de la Figure 4.4.4.

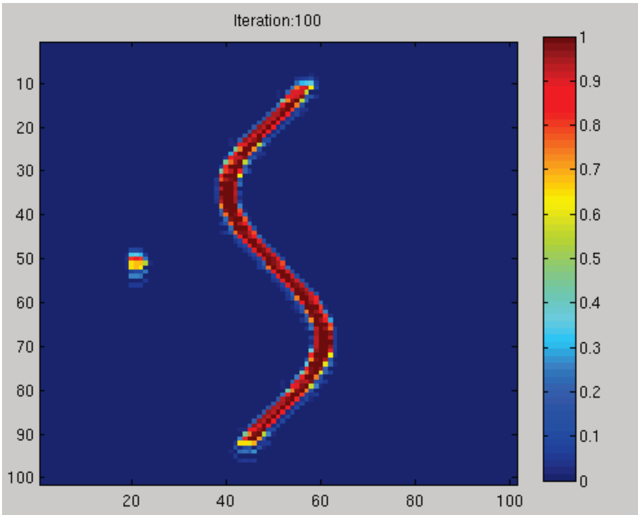

FIGURE 4.4.4 – Cas anisotrope

Contrairement au cas isotrope, il est possible dans le cas anisotrope de *séparer* l'intensité lumineuse de la boule de celle du tube. De plus, cet écart s'accentue si leur rayon diminue. Pour une section égale à 1 pixel, toujours avec les mêmes paramètres, nous obtenons les résultats de la Figure 4.4.7.

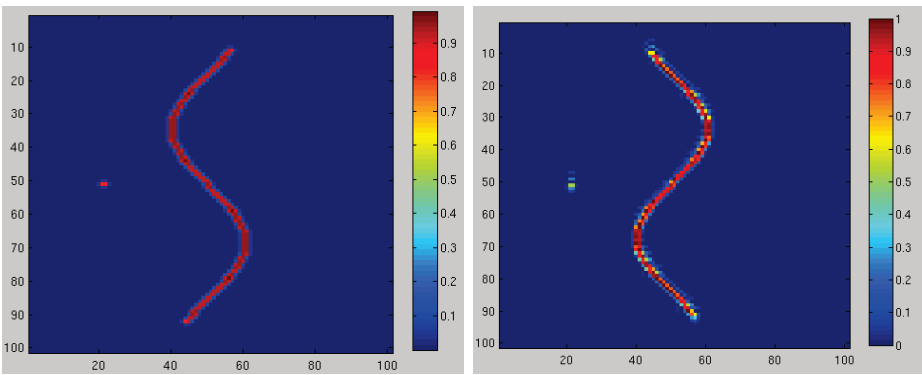

FIGURE 4.4.5 – Diffusion isotrope et anisotrope

C'est cette divergence entre l'intensité d'une boule et celle d'un tube qu'il s'agit d'exploiter pour segmenter l'image. Dans le cas d'une image bruitée, avec une section toujours égale à 1 pixel, nous pouvons séparer, par un seuillage adapté, les points des filaments.

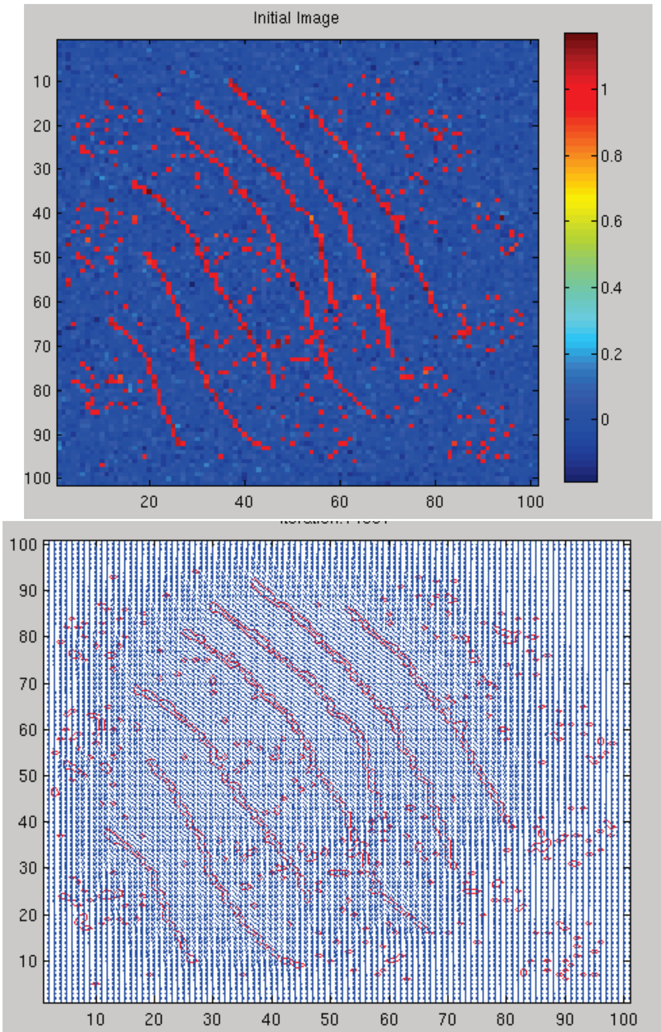

FIGURE 4.4.6 – Image initiale et champ **c**

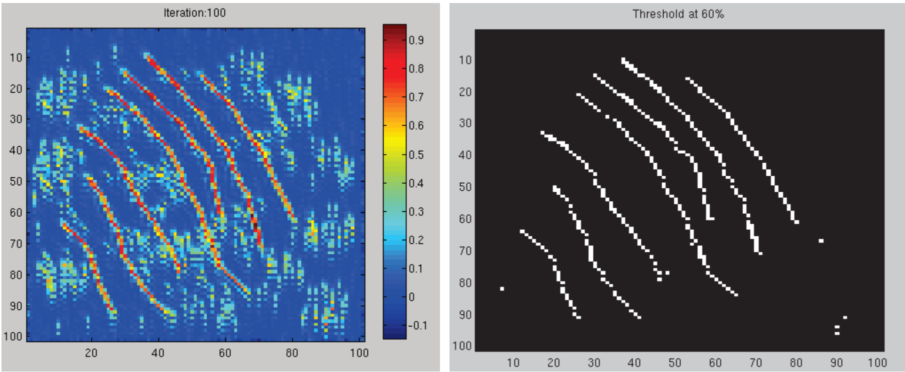

FIGURE 4.4.7 – Itération 100 et seuillage à 60%

Dans cet exemple, la hauteur de seuillage est déterminée empiriquement. Cependant, sur le modèle de l'article [BV14], il est possible d'exhiber les *profils types* d'une boule et d'un tube pour pouvoir déterminer la hauteur de seuillage de manière automatique.

Conclusion et perspectives

Dans le volet théorique de ce travail, nous avons puisé dans les propriétés remarquables de la fonctionnelle de Mumford-Shah pour construire et étudier notre modèle. Ces propriétés ont permis d'asseoir notre modèle sur des bases théoriques saines. Cela constitue pour nous une motivation importante pour participer à la continuation du travail théorique sur le sujet de la fonctionnelle de Mumford-Shah. En particulier, le couplage avec la géométrie des objets à segmenter doit être approfondi : nous pourrons ainsi aborder le problème des bifurcations (puisque nous cherchons un arbre vasculaire) que nous n'avons pas envisagé dans cette étude. Le passage par un terme d'anisotropie plus général qu'une métrique Riemannienne pourra être envisagé.

Sur le plan numérique, l'introduction de ce modèle nous a conduit à des algorithmes de diffusion anisotrope (voir l'équation (4.2.1)).

$$\left\{ \begin{array}{rcll} -\mathrm{div}(\mathbf{M}\nabla p) + \tilde{f}(p) & = & & \text{pour } x \in \Omega, \\ \dfrac{\partial p}{\partial \mathbf{n}} & = & 0 & \text{pour } x \in \partial\Omega. \end{array} \right.$$

Les algorithmes produits par ce modèle sont proches de celui de Perona-Malik [PM90]. Dans cette approche, les auteurs avaient considéré une équation d'évolution du type

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \mathrm{div}(c\nabla p)$$

où p et c dépendent de la position et du temps. Nous avons fait le lien entre les paramètres de ce modèle et les caractéristiques géométriques du tube (rayon, élongation et courbure). Il reste à proposer un seuillage automatique de l'image obtenue en fonction de ces contraintes géométriques.

Enfin, il nous reste encore à implémenter ces techniques dans le contexte d'une image non binaire et tridimensionnelle et à les tester sur des images réelles issues de l'acquisition IRM faite dans le laboratoire partenaire (CBM), ce que nous n'avons pas pu faire faute de temps.

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David VICENTE

Modèles de Mumford-Shah pour la détection de structures fines en image

Résumé :

Cette thèse est une contribution au problème de détection de fines structures tubulaires dans une image 2-D ou 3-D. Nous avons plus précisément en vue le cas des images angiographiques. Celles-ci étant bruitées, les vaisseaux ne se détachent pas nettement du reste de l'image, la question est donc de segmenter avec précision le réseau sanguin. Le cadre théorique de ce travail est le calcul des variations et en particulier l'énergie de Mumford-Shah. Cependant, ce modèle n'est adapté qu'à la détection de structures volumiques étendues dans toutes les directions de l'image. Le but de ce travail est donc de construire une énergie qui favorise les ensembles qui ne sont étendus que dans une seule direction, ce qui est le cas de fins tubes. Pour cela, une nouvelle inconnue est introduite, une métrique Riemannienne, qui a pour but la détection de la structure géométrique de l'image. Une nouvelle formulation de l'énergie de Mumford-Shah est donnée avec cette nouvelle métrique. La preuve de l'existence d'une solution au problème de la minimisation de l'énergie est apportée. De plus, une approximation par gamma-convergence est démontrée, ce qui permet ensuite de proposer et de mettre en œuvre une implémentation numérique.

Mots clés : angiographie, segmentation, modèle de Mumford-Shah, anisotropie, SBV, approximations par gamma-convergence de type Modica-Mortola et Ambrosio-Tortorelli, contenu anisotrope de Minkowski, régularité Ahlfors

Mumford-Shah model for detection of fine structures in image processing

Abstract :

This thesis is a contribution to the fine tubular structures detection problem in a 2-D or 3-D image. We are specifically interested in the case of angiographic images. The vessels are surrounded by noise and then the question is to segment precisely the blood network. The theoretical framework of our work is the calculus of variations and we focus on the Mumford-Shah energy. Initially, this model is adapted to the detection of volumetric structures extended in all directions of the image. The aim of this study is to build an energy that favors sets which are extended in one direction, which is the case of fine tubes. Then, we introduce a new unknown, a Riemannian metric, which captures the geometric structure at each point of the image and we give a new formulation of the Mumford-Shah energy adapted to this metric. The complete analysis of this model is done: we prove that the associated problem of minimization is well posed and we introduce an approximation by gamma-convergence more suitable for numerics. Eventually, we propose numerical experimentations adapted to this approximation.

Keywords: angiography, segmentation, Mumford-Shah model, anisotropy, SBV, gamma-convergence, Modica-Mortola, Ambrosio-Tortorelli, anisotropic Minkowski content, Ahlfors regularity

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